DIFFERENTIAL EQUATIONS

EXERCISE 2.14,2.15

Problems solved by;

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General Sol.

① $y^{iv} - 16y = 0$.

$\Rightarrow \lambda^4 - 16 = 0 \Rightarrow \lambda^2 = \pm 4 \Rightarrow \lambda = \lambda = \pm 2, \pm 2i$.

$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + A\cos 2x + B\sin 2x$.
Ans

② $y''' + 9y'' + 27y' + 27 = 0$

$\Rightarrow \lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$.

Its one root is.

$(\lambda)^3 + 3(\lambda)^2(3) + 3(\lambda)(3)^2 + (3)^3 = 0$.

$(\lambda + 3)^3 = 0$.

$\Rightarrow \lambda = -3, -3, -3$.

$\therefore y = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 x^2 e^{-3x}$ Ans.

③ $y^{iv} - 2y'' + y = 0$

$\lambda^4 - 2\lambda^2 + 1 = 0$.

$\Rightarrow \lambda^4 - \lambda^2 - \lambda^2 + 1 = 0 \Rightarrow \lambda^2(\lambda^2 - 1) - 1(\lambda^2 - 1) = 0$.

$\Rightarrow (\lambda^2 - 1)(\lambda^2 - 1) = 0 \Rightarrow \lambda = \pm 1, \pm 1$.

$\therefore y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-x} + c_4 x e^{-x}$

④ $y^{iv} + 2y'' + y = 0 \Rightarrow \lambda^4 + 2\lambda^2 + 1 = 0$. —①

Sol. $\lambda^2 = u$. $\therefore \lambda^4 = u^2$.

So ① $\Rightarrow u^2 + 2u + 1 = 0$

$\Rightarrow u = \dfrac{-2 \pm \sqrt{4-4}}{2} = \dfrac{-2}{2} = -1, -1$

$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$ & $\lambda^2 = -1 \Rightarrow \lambda = \pm i$

$\therefore y = A_1 \cos x + B_1 \sin x + A_2 x \cos x + B_2 x \sin x$

$\Rightarrow y = (A_1 + A_2 x)\cos x + (B_1 + B_2 x)\sin x$.

⑤ $y''' - 2y'' - y' + 2y = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\Rightarrow \lambda^2(\lambda - 2) - 1(\lambda - 2) = 0$

$\Rightarrow (\lambda^2 - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = \pm 1, 2$.

(6) $y^{iv} + 5y'' + 4y = 0$.

$\Rightarrow \lambda^4 + 5\lambda^2 + 4 = 0$

Q6 $\Rightarrow \lambda^2 = \dfrac{-5 \pm \sqrt{25-16}}{2} = \dfrac{-5 \pm 3}{2} = -1, -4$

$\Rightarrow \lambda^2 = -1, \quad \lambda^2 = -4.$

$\therefore \lambda = \pm i, \quad \lambda = \pm 2i.$

$\therefore y = A\cos x + B\sin x + C\cos 2x + D\sin 2x$

(7) $(D^3 - D^2 - D + 1)y = 0$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0.$

$\Rightarrow \lambda^2(\lambda - 1) - 1(\lambda - 1) = 0 \quad \Rightarrow (\lambda^2 - 1)(\lambda - 1) = 0.$

$\Rightarrow \lambda = \pm 1, \lambda = 1.$

$\therefore y = (c_1 + c_2 x)e^x + c_3 e^{-x}.$ Ans

(8) $(D^3 - 3D + 2)y = 0$

$\Rightarrow \lambda^3 - 3\lambda + 2 = 0. \quad \text{---(1)}$

Now by immagination. $\lambda = 1$ satisfies (1)

i.e $1 - 3(1) + 2 = 0$

$\therefore$

$$\begin{array}{r|l} & \lambda^2 + \lambda - 2 \\ \lambda - 1 & \lambda^3 - 3\lambda + 2 \\ & \underline{\lambda^3 - \lambda^2} \\ & \lambda^2 - 3\lambda + 2 \\ & \underline{\lambda^2 - \lambda} \\ & -2\lambda + 2 \\ & \underline{-2\lambda + 2} \\ & 0 \end{array}$$

$\therefore \lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda^2 + 2\lambda - \lambda - 2 = 0$

$\Rightarrow \lambda(\lambda + 2) - 1(\lambda + 2) = 0$

$\Rightarrow \lambda = 1, \lambda = -2.$

$\therefore \lambda = 1, 1, -2.$

So $y = (c_1 + c_2 x)e^x + c_3 e^{-2x}$ Ans

(9) $(16D^4 - 40D^2 + 9)y = 0$

$\Rightarrow 16\lambda^4 - 40\lambda^2 + 9 = 0.$

$\Rightarrow 16\lambda^4 - 4\lambda^2 - 36\lambda^2 + 9 = 0$

$\Rightarrow 4\lambda^2(4\lambda^2 - 1) - 9(4\lambda^2 - 1) = 0$

$\Rightarrow (4\lambda^2 - 1)(4\lambda^2 - 9) = 0 \quad \Rightarrow \lambda = \pm 1/2, \pm 3/2$

$\Rightarrow y = c_1 e^{x/2} + c_2 e^{-x/2} + c_3 e^{3x/2} + c_4 e^{-3x/2}$.

(10) $(D^3 + 6D^2 + 11D + 6)y = 0$ — (1)

$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$ — (2)

By immagination. $\lambda = -1$ satisfies (2)

ie

$-1 + 6 - 11 + 6 = 0$.

$\therefore$ we divide (2) by $\lambda + 1$ & get other roots as

$$\begin{array}{r|l} & \lambda^2 + 5\lambda + 6 \\ \lambda + 1 & \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\ & \underline{\lambda^3 \pm \lambda^2} \\ & 5\lambda^2 + 11\lambda + 6 \\ & \underline{5\lambda^2 \pm 5\lambda} \\ & 6\lambda + 6 \\ & \underline{6\lambda \pm 6} \end{array}$$

$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$.

$\Rightarrow \lambda^2 + 3\lambda + 2\lambda + 6 = 0$.

$\Rightarrow \lambda(\lambda + 3) + 2(\lambda + 3) = 0$

$\Rightarrow \lambda = -3, -2$.

$\therefore \lambda = -1, -2, -3$.

and $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ Ans

Initial Value Problems:

(11) $y^{iv} = 0$, $y(0) = 1$, $y'(0) = 16$, $y''(0) = -4$, $y'''(0) = 24$

$y^{iv} = 0$.

integrating w.r.t x

$\Rightarrow y''' = c_1 \Rightarrow y'' = c_1 x + c_2 \Rightarrow y' = c_1 \frac{x^2}{2} + c_2 x + c_3$

$\Rightarrow y = \frac{c_1 x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$.

applying initial conditions

$\Rightarrow c_1 = 24$, $c_2 = -4$, $16 = c_3$, $c_4 = 1$.

$\therefore y = 4x^3 - 2x^2 + 16x + 1$

(12) $y''' - 3y'' + 3y' - y = 0$, $y(0) = 2$, $y'(0) = 2$, $y''(0) = 10$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$.

$\Rightarrow (\lambda - 1)^3 = 0$

$\Rightarrow \lambda = 1, 1, 1$

(13) $y''' - y'' - y' + y = 0$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\Rightarrow \lambda(\lambda^2-1) - 1(\lambda^2-1) = 0 \Rightarrow (\lambda-1)(\lambda^2-1) = 0$

Q13 $\Rightarrow \lambda = 1, 1, -1$

$\therefore y = (c_1 + c_2 x)e^x + c_3 e^{-x}$ — ①

$\Rightarrow y' = (c_1 + c_2 x)e^x + e^x c_2 - c_3 e^{-x}$ — ②

$\Rightarrow y'' = e^x(c_1 + c_2 x) + 2e^x c_2 + c_3 e^{-x}$ — ③

Applying initial conditions

③ $\Rightarrow 0 = c_1 + 2c_2 + c_3$ — ④

② $\Rightarrow 1 = c_1 + c_2 - c_3$ — ⑤

③ $\Rightarrow 2 = c_1 + c_3$ — ⑥

Adding ④ and ⑤ $\Rightarrow 1 = 2c_1 + 3c_2$ — ⑦

Adding ⑤ and ⑥ $\Rightarrow 3 = 2c_1 + c_2$. — ⑧

Subtracting ⑦ and ⑧ $\Rightarrow 2 = -2c_2 \Rightarrow c_2 = 1$

$\therefore 2c_1 - 1 = 3 \Rightarrow c_1 = 4/2 = 2$.

$\therefore c_3 = 2 - 2 = 0$.

So $y = (2 - x)e^x$ Ans

(14) $(D^4 - 1)y = 0$, $y(0) = 1$, $y'(0) = 7$, $y''(0) = -1$, $y'''(0) = 7$

$\lambda^4 - 1 = 0 \Rightarrow \lambda^2 = \pm 1 \Rightarrow \lambda = \sqrt{\pm 1} = \pm 1, \pm i$.

$\therefore y = c_1 e^x + c_2 e^{-x} + A\cos x + B\sin x$ — ①

$\Rightarrow y' = c_1 e^x - c_2 e^{-x} - A\sin x + B\cos x$ — ②

$\Rightarrow y'' = c_1 e^x + c_2 e^{-x} - A\cos x - B\sin x$ — ③

$\Rightarrow y''' = c_1 e^x - c_2 e^{-x} + A\sin x - B\cos x$ — ④

Applying initial conditions.

④ $\Rightarrow 7 = c_1 - c_2 - B$ — ⑤

③ $\Rightarrow -1 = c_1 + c_2 - A$ — ⑥

② $\Rightarrow 7 = c_1 - c_2 + B$ — ⑦

① $\Rightarrow -1 = c_1 + c_2 + A$ — ⑧

Adding ⑤ and ⑦ Adding ⑥ and ⑧

$6 = 2c_1 - 2c_2$ — ⑨ $-2 = 2c_1 + 2c_2$ — ⑩

$\Rightarrow 4 = 4c_1 \Rightarrow c_1 = 1.$

$\therefore 2c_2 = -4 \Rightarrow c_2 = -2.$

(9)+(10) and so $B = -4.$

and also $A = 0.$

$\therefore y = e^{x} - 2e^{-x} - 4\sin x.$ Ans

(15) $(D^3 + 5D^2 - D - 5)y = 0, \quad y(0) = 5, \; y'(0) = 0, \; y''(0) = 12$

$\lambda^3 + 5\lambda^2 - \lambda - 5 = 0$

$\Rightarrow \lambda(\lambda^2 - 1) + 5(\lambda^2 - 1) = 0.$

$\Rightarrow (\lambda + 5)(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = -5, -1, +1.$

$\therefore y = c_1 e^{-5x} + c_2 e^{-x} + c_3 e^{x}.$

$\Rightarrow y' = -5c_1 e^{-5x} - c_2 e^{-x} + c_3 e^{x}.$

$\Rightarrow y'' = 25c_1 e^{-5x} + c_2 e^{-x} + c_3 e^{x}$

$\Rightarrow y''' = -125c_1 e^{-5x} - c_2 e^{-x} + c_3 e^{x}.$

Applying initial conditions

$125 = 25c_1 + c_2 + c_3. \quad —(1)$

& $0 = -5c_1 - c_2 + c_3. \quad —(2)$

$5 = c_1 + c_2 + c_3. \quad —(3)$

adding (1) and (2)

$\Rightarrow 125 = 20c_1 + 2c_3. \quad —(4)$

adding (2) and (3)

$\Rightarrow 5 = -4c_1 + 2c_3 \quad —(5)$

Subtracting (5) from (4)

$\Rightarrow 120 = 24c_1 \Rightarrow c_1 = \frac{120}{24} = \frac{60}{12} = \frac{30}{6} = 5$

$\therefore c_3 = +\frac{25}{2}.$

$\therefore c_2 = \frac{-25}{2} = -\frac{25}{2}$

$\therefore y = 5e^{-5x} + 25\left(\frac{e^{x} - e^{-x}}{2}\right)$

$x^4+10x^2+9.$ $\quad y'''(0)=0$

$\Rightarrow \lambda^4+9\lambda^2+\lambda^2+9=0 \Rightarrow \lambda^2(\lambda^2+9)+1(\lambda^2+9)=0$

$\Rightarrow (\lambda^2+1)(\lambda^2+9)=0$

$\Rightarrow \lambda = \pm 3i, \pm i.$

$\therefore y = A\cos 3x + B\sin 3x + C\cos x + D\sin x.$

Q16 $\Rightarrow y' = -3A\sin 3x + 3B\cos 3x - C\sin x + D\cos x$

$\Rightarrow y'' = -9A\cos 3x - 9B\sin 3x - C\cos x - D\sin x$

$\Rightarrow y''' = 27A\sin 3x - 27B\cos 3x + C\sin x - D\cos x.$

Applying initial conditions

$0 = A + C \quad -(1) \qquad 32 = -9A - C \quad -(3)$

$0 = 3B + D \quad -(2) \qquad 0 = -27B - D \quad -(4)$

adding (1) and (3).

$32 = -8A \Rightarrow A = -4$ and $C = +4.$

adding (2) and (4).

$\Rightarrow 0 = 24B \Rightarrow B = 0$ & $D = 0.$

$\therefore y = -4\cos 3x + 4\cos x.$ (Ans)

(17) $(D^4+4D^3+8D^2+8D+4)y=0$

$y(0)=1, y'(0)=0, y''(0)=-2, y'''(0)=2$

sol we have

$\lambda^4 + 4\lambda^3+8\lambda^2+8\lambda+4=0.$

$\lambda^4+8\lambda^2+4+4\lambda^3+8\lambda=0.$

$\Rightarrow \lambda^4 + .$

$\lambda^4+4\lambda^3+8\lambda^2+8\lambda+4$

$=\lambda^4+8\lambda^2+4+4\lambda^3+8\lambda$

$=\lambda^4+4\lambda^2+4)+4\lambda^3+4\lambda^2+8\lambda$

$=(\lambda+2)^2+4\lambda(\lambda^2+\lambda+2)$

$\lambda^4+4\lambda^3+8\lambda^2+8\lambda+4$

$=\lambda^4+4\lambda^3+4\lambda^2+4\lambda^2+8\lambda+4$

$=\lambda^2(\lambda^2+4\lambda+4)+4(\lambda^2+2\lambda+1)$

$=\lambda^2(\lambda+2)^2+4(\lambda+1)^2$

$(x-a)(x-b)(x-c)(x-d)=0$

$(x^2-(a+b)x+ab)(x^2-(c+d)x+cd)=0$

$x^4-(a+b+c+d)x^3+(ad+bc+ab+cd)x^2$

EX : 2.15

(13) $y'''-4y'=10\cos x+5\sin x,\quad y(0)=3,\ y'(0)=-2,\ y''(0)=-1.$

for y_h: $\lambda^3-4\lambda=0 \Rightarrow \lambda(\lambda^2-4)=0.$

$\Rightarrow \lambda=0,\ +2,\ -2$

$\therefore y_h = c_1e^0+c_2e^{2x}+c_3e^{-2x}.$

Also

$$W=\begin{vmatrix}1 & e^{2x} & e^{-2x}\\ 0 & 2e^{2x} & -2e^{-2x}\\ 0 & 4e^{2x} & 4e^{-2x}\end{vmatrix}$$

$\Rightarrow W = 8e^{-2x}+8e^{-2x}=16e^{-2x}.$

$$W_1=\begin{vmatrix}0 & e^{2x} & e^{-2x}\\ 0 & 2e^{2x} & -2e^{-2x}\\ 1 & 4e^{2x} & 4e^{-2x}\end{vmatrix}=-4$$

$$W_2=\begin{vmatrix}1 & 0 & e^{-2x}\\ 0 & 0 & -2e^{-2x}\\ 0 & 1 & 4e^{-2x}\end{vmatrix}=+2e^{-2x}.$$

$$W_3=\begin{vmatrix}1 & e^{2x} & 0\\ 0 & 2e^{2x} & 0\\ 0 & 4e^{2x} & 1\end{vmatrix}=2e^{2x}.$$

$\therefore y_p = y_1\int\frac{W_1}{W}r(x)\,dx+y_2\int\frac{W_2}{W}r(x)\,dx+y_3\int\frac{W_3}{W}r(x)\,dx$

$\Rightarrow y_p = \int\frac{-4}{16e^{-2x}}(10\cos x+5\sin x)\,dx+e^{2x}\int\frac{2e^{-2x}}{16e^{-2x}}(10\cos x+5\sin x)\,dx$

$+\ e^{-2x}\int\frac{2e^{2x}}{16e^{-2x}}(10\cos x+5\sin x)\,dx$

$= \frac{-1}{4}\int e^{2x}(10\cos x + 5\sin x)\,dx$

$x^3D^3y + xDy - y = x^2$

or $\cancel{x^3D^3}(\Delta(\Delta-1)(\Delta-2) + \Delta - 1)y = e^{2t}$

or $(\Delta^3 - 3\Delta^2 + 2\Delta + \Delta - 1)y = e^{2t}$

or $(\Delta^3 - 3\Delta^2 + 3\Delta - 1)y = e^{2t}$

$y = A\cos x + B\sin x$, $y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$, $y''' = A\sin x - B\cos x$

$y''' - 4y' = 10\cos x + 5\sin x$

$A\sin x - B\cos x - 4(-A\sin x + B\cos x) = 10\cos x + 5\sin x$

or $5A\sin x - 5B\cos x = 5\sin x + 10\cos x$

$A = 1$, $B = -2$

$y_p = \cos x - 2\sin x$

$y = y_h + y_p = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \cos x - 2\sin x$

$y(0) = c_1 + c_2 + c_3 + 1 = -1$

$y' = 2c_2 - 2c_3 = 0 \qquad c_2 = c_3$

$y'' = 4c_2 + 4c_3 = 0 \qquad c_2 = -c_3 \quad c_2 = c_3 = 0$

(10)

$x^3 y''' + xy' - y = x^2$, $y(1) = 1$, $y'(1) = 3$, $y''(1) = 3$

$x = e^t \Rightarrow t = \ln x$. $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

So $x\frac{dy}{dx} = \frac{dy}{dt}$ or $xDy = \Delta y$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{1}{x}\left(\frac{1}{x}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\frac{dx}{dt}\right)$

Let $x = e^t \Rightarrow t = \ln x$

$\frac{dx}{dt} = e^t = x$ and $\frac{dt}{dx} = \frac{1}{x}$

Now $Dy = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x} = \frac{1}{x}\frac{dy}{dt} = \frac{1}{x}\Delta y$

So, $Dy = \frac{1}{x}\cdot\Delta y$ or $\Delta y = xDy \quad \text{---}(1)$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\frac{1}{x}$

$$= \left(\frac{1}{x}\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\frac{1}{x}\right)\cdot\frac{dy}{dt}\right)\cdot\frac{1}{x} = \frac{1}{x}\left(\frac{1}{x}\frac{d^2y}{dt^2} - \frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \cancel{= \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt} - y\right)}$$

$$= \frac{1}{x^2}\left(\Delta^2 y - \Delta y\right) = \frac{1}{x^2}\left(\Delta^2 - \Delta\right)y$$

So, $D^2y = \frac{1}{x^2}(\Delta^2 - \Delta)y$

or $x^2D^2y = (\Delta^2 - \Delta)y = \Delta(\Delta - 1)y \quad -(2)$

Now $D^3y = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{dt}{dx}$

$$= \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{1}{x} = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right)$$

$$= \frac{1}{x}\left(\frac{1}{x^2}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right) - \frac{2}{x^3}\frac{d}{dt}(x)\cdot\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right)$$

$$= \frac{1}{x}\left(\frac{1}{x^2}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right) - \frac{2x}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right)$$

$$\boxed{\frac{d}{dt}(x) = \frac{d}{dt}(e^t) = e^t = x}$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - 2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right)$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) = \frac{1}{x^3}\left(\Delta^3 y - 3\Delta^2 y + 2\Delta y\right)$$

& Thus, $D^3y = \frac{1}{x^3}(\Delta^3 - 3\Delta^2 + 2\Delta)y$

or $x^3D^3y = (\Delta^3 - 3\Delta^2 + 2\Delta)y$

$$= \Delta(\Delta^2 - 3\Delta + 2)y$$

$$x^3Dy = \Delta(\Delta - 1)(\Delta - 2)y \quad -(3)$$

Thus, we have $xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1)$, $x^3D^3 = \Delta(\Delta - 1)(\Delta - 2)$,

$x^4D^4 = \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3)$ and so on

upto $x^nD^n = \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3)\dots(\Delta - (n-1))$

where $D = \frac{d}{dx}$, $\Delta = \frac{d}{dt}$ and $n \in Z^+$.

$1-6+11-6=0$ 11

(ii) $x^3y'''-3x^2y''+6xy'-6y=24x^5$

$y(1)=1,\ y'(1)=3,\ y''(1)=14.$

for y_h.

$m(m-1)(m-2)-3m(m-1)+6m-6=0$

$\Rightarrow (m^2-m)(m-2)-3(m^2-m)+6m-6=0.$

A.E. $\Rightarrow m^3-3m^2+2m-3m^2+3m+6m-6=0.$

$\Rightarrow m^3-6m^2+11m-6=0.$

Its one root is $m=1$.

because for $m=1$.

$1-6+11-6=12-12=0.$

other roots can be obtained by division by $m-1$

i.e.

$$\begin{array}{r|l} & m^2-5m+6 \\ m-1 & m^3-6m^2+11m-6 \\ & \underline{\mp m^3 \pm m^2} \\ & -5m^2+11m \\ & \underline{\mp 5m^2 \pm 5m} \\ & 6m-6 \\ & \underline{\mp 6m \pm 6} \\ & \times \end{array}$$

we have

$m^2-5m+6=0.$

$\Rightarrow m^2-3m-2m+6=0$

$\Rightarrow (m-3)(m-2)=0$

$\Rightarrow m=2,\ m=3.$

$\therefore y_h=c_1x+c_2x^2+c_3x^3.$

Now

$$W=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$\Rightarrow W=\begin{vmatrix} 0 & x^2 & 2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \quad -xR_2+R_1$$

$\therefore W=6x^3-4x^3=+2x^3.$

$x \cdot 2x^2\ 3x^3.$

$x \quad x^2 \quad x^3$

$0 \quad x^2 \quad 2x^3.$

$$\begin{vmatrix} x & x^2 & x^3 \\ 0 & x^2 & 5x^3 \\ 0 & 2 & 6x \end{vmatrix}$$

$x(6x^3-10x^3)$

$\Rightarrow x(-4x^3)$

$=-4x^4.$

$$W_1=\begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix}=x^4.$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

$$y_p = x\int \frac{x^4}{2x^3}\cdot 24x^2\,dx + x^2\int \frac{2x^3}{2x^3}\cdot 24x^2\,dx + x^3\int \frac{x^2}{2x^3}\cdot 24x^2\,dx$$

$$\Rightarrow y_p = 12x\int x^3\,dx + 24x^2\int x^2\,dx + 12x^3\int x\,dx.$$

$$\Rightarrow y_p = 12x\left(\frac{x^4}{4}\right) + 24x^2\left(\frac{x^3}{3}\right) + 12x^3\left(\frac{x^2}{2}\right)$$

$$\Rightarrow y_p = 3x^5 + 8x^5 + 6x^5 = 17x^5.$$

$$\therefore y = c_1x + c_2x^2 + c_3x^3 + 17x^5.$$

$$\Rightarrow y' = c_1 + 2c_2x + 3c_3x^2 + 85x^4.$$

$$\Rightarrow y'' = 2c_2 + 6c_3x + 340x^3$$

$$\Rightarrow y''' = 6c_3 + 1020x^2$$

Applying Initial conditions.

$$1 = c_1 + c_2 + c_3 + 17 \Rightarrow c_1 + c_2 + c_3 = -16. \quad — (1)$$

$$3 = c_1 + 2c_2 + 3c_3 + 85 \Rightarrow c_1 + 2c_2 + 3c_3 = -82. \quad — (2)$$

$$14 = 2c_2 + 6c_3 + 340 \Rightarrow 2c_2 + 6c_3 = -326. \quad — (3)$$

subtracting (1) from (2) $\Rightarrow c_2 + 3c_3 = -163$

$$\Rightarrow c_2 + 2c_3 = -66 \quad — (4)$$

subtracting (4) from (3)

$$\Rightarrow c_3 = -97$$

Hence $c_2 = 128$.

& $c_1 = -47$.

 Solved by: Umair Asghar

(9) $y''' - 3y'' + 3y' + y = e^{-x}\sin x$

$\lambda^3 - 3\lambda^2 + 3\lambda + 1 = 0$

$\Rightarrow \cancel{\lambda^2(\lambda-3)}$

aq $\cancel{\lambda(\lambda^2+3)-3(}$

$(\lambda+1)^3 = 0 \quad \lambda = -1, -1, -1.$

$y_h = (c_1 + c_2 x + c_3 x^2)e^{-x}.$

$y_p = x e^{-x}(A\cos x + B\sin x)$

$y_p = Ae^{-x}\cos x.$

$y(0)=0,\ y'(0)=6,\ y''(0)=0,\ y'''(0)=-26$

$r^4+10r^2+9=0$

$(r^2+9)(r^2+1)=0$

$r^2=-9,\ r^2=-1$

$r=\pm 3i,\ \pm i$

$y_h = c_1\cos 3x + c_2\sin 3x + c_3\cos x + c_4\sin x$

$y_p = A\cosh x + B\sinh x$

$y_p' = A\sinh x + B\cosh x$

$y_p'' = A\cosh x + B\sinh x$

$y_p''' = A\sinh x + B\cosh x$

$y_p^{(iv)} = A\cosh x + B\sinh x$

$(A\cosh x + B\sinh x) + 10(A\cosh x + B\sinh x)$
$+9(A\cosh x + B\sinh x) = 40\sinh x$

or $20A\cosh x + 20B\sinh x = 40\sinh x$

$\Rightarrow A=0,\ B=2$

$y_p = 2\sinh x$

$y'_p = 2\cosh x,\ y''_p = 2\sinh x,\ y'''_p = 2\cosh x,\ y^{iv}_p = 2\sinh x$

$2\sinh x + \underline{20}\sinh x + \underline{18}\sinh x = 40\sinh x$

$y = c_1\cos 3x + c_2\sin 3x + c_3\cos x + c_4\sin x + 2\sinh x$

$\Rightarrow y' = -3c_1\sin 3x + 3c_2\cos 3x - c_3\sin x + c_4\cos x + 2\cosh x$

$\Rightarrow y'' = -9c_1\cos 3x - 9c_2\sin 3x - c_3\cos x - c_4\sin x + 2\sinh x$

$\Rightarrow y''' = 27c_1\sin 3x$

$\Rightarrow y^{iv} = +81c_1\cos 3x + 81c_2\sin 3x + c_3\cos x + c_4\sin x + 2\sinh x$

$+ 2\sinh x - 40c_1\cos 3x - 90c_2\sin 3x - 10c_3\cos x$

$-10c_4\sin x + 20\sinh x + 9c_1\cos 3x + 9c_2\sin 3x$

$+ 9c_3\cos x + 9c_4\sin x + 18\sinh x.$

$y_p = Ae^{3x} + Be^{-3x}$

$\Rightarrow y'_p = 3Ae^{3x} - 3Be^{-3x}$

$\Rightarrow y''_p = 9Ae^{3x} + 9Be^{-3x}$

$y'''_p = 27Ae^{3x} - 27Be^{-3x}$

$y''''_p = 81Ae^{3x} + 81Be^{-3x}$

$81Ae^{3x} + 18Be^{-3x} + 90Ae^{3x} + 90Be^{-3x} + 9Ae^{3x} + 9Be^{-3x} = 20e^{3x}$

$y = Ae^{x} + Be^{-x}$

$y' = Ae^{x} - Be^{-x}$

$y'' = Ae^{x} + Be^{-x}$

$y''' = Ae^{x} - Be^{-x}$

$y'''' = Ae^{x} + Be^{-x}$

$Ae^{x} + Be^{-x} + 10Ae^{x} + 10Be^{-x} + 9Ae^{x} + 9Be^{-x}$

$20Ae^{x} + 20Be^{-x} = 20e^{x} - 20e^{-x}$

$A = 1, \quad B = -1$

$\therefore y_p = e^{x} - e^{-x} = 2\sinh x$

y

$\Rightarrow y_h = Ae^{2x} + Be^{-2x} + C\cos 2x + D\sin 2x.$

Q13 $y_p = M\cos x + N\sin x.$

$\Rightarrow y'_p = -M\sin x + N\cos x$

$\Rightarrow y''_p = -M\cos x - N\sin x.$

$\Rightarrow y'''_p = M\sin x - N\cos x$

$\Rightarrow y^{IV}_p = M\cos x + N\sin x.$

$\Rightarrow$

~~$M\cos x + N\sin x$~~ $- 4$

(15) $y''' - 4y' = 10\cos x + 5\sin x.$ — (1)

$\Rightarrow \lambda^3 - 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) \Rightarrow \lambda = 0, 2, -2.$

$\Rightarrow y_h = c_1 + c_2 e^{2x} + c_3 e^{-2x}.$

$y_p = A\cos x + B\sin x.$

$\Rightarrow y'_p = -A\sin x + B\cos x.$

$\Rightarrow y''_p = -A\cos x - B\sin x.$

$\Rightarrow y'''_p = A\sin x - B\cos x$

(1) $\Rightarrow A\sin x - B\cos x + 4A\sin x - 4B\cos x = 10\cos x + 5\sin x$

$\Rightarrow +5A\sin x - 5B\cos x = 10\cos x + 5\sin x$

$\Rightarrow 5A = 5 \Rightarrow A = 1, B = -2.$

$\therefore y_p = \sin x - 2\cos x.$

$\therefore y_h = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \cos x - 2\sin x.$

$y'_h = 2c_2 e^{2x} - 2c_3 e^{-2x} - \sin x - 2\cos x$

$y''_h = 4c_2 e^{2x} + 4c_3 e^{-2x} - \cos x + 2\cos x$

$c_1 + c_2 + c_3 + 1 = 3.$

$\Rightarrow c_1 + c_2 + c_3 = 2.$

$2c_2 - 2c_3 - 2 = -2 \Rightarrow 2c_2 - 2c_3 = 0$

$\Rightarrow 4c_2 + 4c_3 - 1 = -1$ $\quad 4c_2 - 4c_3 = -6$

$\Rightarrow 2(2c_2 + 2c_3) = -2$ $\quad 4c_2 + 4c_3 = -2$

$A\sin x - B\cos x + 4A\sin x - 4B\cos x = 10\cos x + 5\sin x$

$\Rightarrow \underline{\cos x - 2\sin x}$

$2c_2 + 2c_3 = -1$

$2c_2 - 2c_3 = 0$

$4c_2 = -1 \Rightarrow c_2 = -1/4$

$c_3 = -\frac{1}{2} = -2c_3$

$\Rightarrow c_3 = 1/4$

$-\frac{1}{4} + \frac{1}{4} + c_1 = 5$

$\Rightarrow c_1 = 2$

$\Rightarrow 2 - \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \sin x - 2\cos x$

$\Rightarrow y_p = 2 - \frac{1}{2}\sinh x + \sin x - 2\cos$

$y_p = \underline{\cos x - 2\sin x}$

$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \cos x - 2\sin x$

$y' = 2c_2 e^{2x} - 2c_3 e^{-2x} - \sin x - 2\cos x$

$y'' = 4c_2 e^{2x} + 4c_3 e^{2x} - \cos x + 2\sin x$

$-1 = 4c_2 + 4c_3 - 1$

$\Rightarrow 4c_2 + 4c_3 = 0 \qquad \Rightarrow c_2 + c_3 = 0$

$-2 = 2c_2 - 2c_3 - 2 \Rightarrow 2c_2 - 2c_3 = 0 \Rightarrow c_2 \cdot c_3 = 0$

$c_1 + c_2 + c_3 + 1 = 3 \Rightarrow \underline{c_1 = 2}$