DIFFERENTIAL EQUATIONS

EXERCISE 2.13

Problems solved by;

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Ex - 2-13.

Prove that the given functions form the basis of the corresponding given equation. Then solve initial value problem.

① $1, x, x^2, x^3, \quad y^{IV} = 0,$
$y(0)=1, \; y'(0)=0, \; y''(0)=-1, \; y'''(0)=30.$

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

expanding by column ①

$$W = 1 \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & 2 & 6x \\ 0 & 0 & 6 \end{vmatrix}$$

expanding by column ① again

$$W = 1 \begin{vmatrix} 2 & 6x \\ 0 & 6 \end{vmatrix} = 12 \neq 0$$

hence they form the basis.

Now $y^{IV} = 0$

$\Rightarrow y''' = c_1 \Rightarrow y'' = c_1 x + c_2 \Rightarrow y' = \frac{c_1 x^2}{2} + c_2 x +$

$\Rightarrow y = \frac{c_1}{6} x^3 + \frac{c_2}{2} x^2 + c_3 x + c_4.$

Appling initial conditions.

$1 = c_4, \quad c_3 = 0, \quad c_2 = -1, \quad c_1 = 30.$

$\therefore y = 5x^3 + (-\frac{1}{2})x^2 + 1$

$\Rightarrow y = 5x^3 - \frac{1}{2}x^2 + 1.$

$y(0)=4,\ y'(0)=-13,\ y''(0)=46.$

Sol

Q3

$$W=\begin{vmatrix} e^{-3x} & xe^{-3x} & x^2e^{-3x} \\ -3e^{-3x} & e^{-3x}-3xe^{-3x} & 2xe^{-3x}-3x^2e^{-3x} \\ 9e^{-3x} & -6e^{-3x}+9xe^{-3x} & 2e^{-3x}-12xe^{-3x}+9x^2e^{-3x} \end{vmatrix}$$

$$\Rightarrow W = e^{-9x}\begin{vmatrix} 1 & x & x^2 \\ -3 & 1-3x & 2x-3x^2 \\ 9 & -6+9x & 2-12x+9x^2 \end{vmatrix}$$

$$\Rightarrow W = e^{-9x}\begin{vmatrix} 1 & 0 & 0 \\ -3 & 1 & 2x \\ 9 & -6 & 2-12x \end{vmatrix} \quad \begin{matrix} -C_1x+C_2 \\ -C_1x^2+C_3 \end{matrix}$$

expanding by row (1)

$$\Rightarrow W = e^{-9x}\begin{vmatrix} 1 & 2x \\ -6 & 2-12x \end{vmatrix} = e^{-9x}(2-12x+12x)$$

$\Rightarrow W = 2e^{-9x} \neq 0.$

hence these functions form the basis of the given eq.

Now

$y'''+9y''+27y'+27y=0$

we have

$\lambda^3+9\lambda^2+27\lambda+27=0$ —— (1)

$\Rightarrow$ ~~$\lambda(\lambda^2+27$~~

one root is $\lambda=-3$

we divide (1) by $\lambda+3$

$$\begin{array}{r|l} & \lambda^2+6\lambda+9 \\ \lambda+3 & \lambda^3+9\lambda^2+27\lambda+27 \\ & \underline{\lambda^3+3\lambda^2} \\ & 6\lambda^2+27\lambda \\ & \underline{6\lambda^2+18\lambda} \\ & 9\lambda+27 \end{array}$$

we get

$\lambda^2+6\lambda+9=0$

$\Rightarrow \lambda^2+3\lambda+3\lambda+9=0$

$\Rightarrow \lambda(\lambda+3)+3(\lambda+3)=$

$\Rightarrow \lambda=-3,-3.$

& $y''(0) = -11$

Q2

$$W = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$\Rightarrow W = e^{2x}\begin{vmatrix} 0 & 1 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix} \qquad \underline{C_1 - C_2}.$$

expanding by column ①.

$$\Rightarrow W = 2e^{2x}\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 2e^{2x}(4-1) = 6e^{2x} \neq 0$$

hence they form the Basis.

Now $y''' - 2y'' - y' + 2y = 0$. The characteristic eq is

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\Rightarrow \lambda^3 - \lambda - 2\lambda^2 + 2 = 0$

$\Rightarrow \lambda(\lambda^2 - 1) - 2(\lambda^2 - 1) = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 2, 1, -1.$

$\therefore y_h = c_1 e^{2x} + c_2 e^x + c_3 e^{-x}$. —①

$\Rightarrow y'_h = 2c_1 e^{2x} + c_2 e^x - c_3 e^{-x}$. —②

$\Rightarrow y''_h = 4c_1 e^{2x} + c_2 e^x + c_3 e^{-x}$. —③

Applying initial conditions

①$\Rightarrow -2 = c_1 + c_2 + c_3$.

②$\Rightarrow -5 = 2c_1 + c_2 - c_3$ } adding $\Rightarrow -7 = 3c_1 + 2c_2$

③$\Rightarrow -11 = 4c_1 + c_2 + c_3$ } adding $\Rightarrow -16 = 6c_1 + 2c_2$ } subtracting

$9 = -3c_1$.

$\Rightarrow c_1 = -3$.

hence $c_2 = \frac{-7+9}{2} = 1$

& $c_3 = 0$

$\therefore y_h = -3e^{2x} + e^x$. Ans

$\cancel{\Rightarrow y'' = 9c_1 e^{-3x} + 9c_2 e^{-3x} + 9c_3 e^{-3x}}$

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x} + 2c_3 x e^{-3x} - 3c_3 x^2 e^{-3x} \quad -(3)$$

$$\Rightarrow y'' = 9c_1 e^{-3x} - 3c_2 e^{-3x} - 3c_2 e^{-3x} + 9c_2 x e^{-3x} + 2c_3 e^{-3x} - 6c_3 x e^{-3x}$$
$$- 6c_3 x e^{-3x} + 9c_3 x^2 e^{-3x} \quad -(4)$$

Applying intial conditions.

$(2) \Rightarrow 4 = c_1.$

$(3) \Rightarrow -13 = -3c_1 + c_2 \Rightarrow c_2 = -13 + 3(4) = -1$

$(4) \Rightarrow 46 = 9c_1 - 6c_2 + 2c_3 \Rightarrow 2c_3 = -9(4) + 6(-1) + 46$

$\Rightarrow c_3 = +2.$

$\therefore y = (4 + (-1x) + (+2x^2))e^{-3x}$ Ans

(4) $1, \cos x, \sin x, \quad y''' + y' = 0; \quad y(0) = 15, y'(0) = 0,$
$y''(0) = -3$

Sol

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$\Rightarrow W = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1 \neq 0$$

Hence these function form a basis.

Now. $y''' + y' = 0$

$\Rightarrow \lambda^3 + \lambda = 0 \Rightarrow \lambda(\lambda^2 + 1) = 0$

$\Rightarrow \lambda = 0, \pm i.$

$\therefore y = c_1 + A\cos x + B\sin x. \quad -(1)$

$\Rightarrow y' = -A\sin x + B\cos x \quad -(2)$

$\Rightarrow y'' = -A\cos x - B\sin x \quad -(3)$

Applying initial conditions

① $\Rightarrow 15 = c_1 + A.$

② $\Rightarrow 0 = B.$

& ③ $\Rightarrow -3 = -A \Rightarrow A = 3. \quad \therefore c_1 = 12.$

so $y = 12 + 3\cos x.$

Check

$y' = -3\sin x \Rightarrow y'' = -3\cos x \Rightarrow y''' = 3\sin x.$

$\therefore 3\sin x - 3\sin x = 0.$ verified

⑤ $e^{x}\cos x,\ e^{x}\sin x,\ e^{-x}\cos x,\ e^{-x}\sin x.$

$(D^4+4)y = 0 \ ; \ y(0)=0,\ y'(0)=2,\ y''(0)=0,\ y'''(0)=4$

Sol

$$\cancel{W = \begin{vmatrix} e^{x}\cos x & e^{x}\sin x & e^{-x}\cos x & e^{-x}\sin x \\ e^{x}\cos x - e^{x}\sin x & e^{x}\sin x + e^{x}\cos x & -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \\ e^{x}\cos x - 2e^{x}\sin x - e^{x}\cos x & e^{x}\sin x + 2e^{x}\cos x - e^{x}\sin x & e^{-x}\cos x + 2e^{-x}\sin x - e^{-x}\cos x & e^{-x}\sin x - 2e^{-x}\cos x + e^{-x}\sin x \end{vmatrix}}$$

$\Rightarrow$ ~~W =~~ ~~cos x sin x~~ ~~cos x~~

$e^{x}\sin x \quad e^{-x}\cos x \quad e^{-x}$

$x = \sqrt{\pm 2i}, \sqrt{-2i}$

$x = \sqrt{2},$ $(D^2+2)(D^2-2)$, $D = \pm 1$

$x^2 = \pm 2i$ $D = \pm^{2i}$

$x^{\frac{1}{2}} = \pm\sqrt{\pm 2i}$ $\Rightarrow$ $\lambda^2 = \pm 2i$

$x = \sqrt{\pm 2i}, -\sqrt{\pm 2i}$ $x^4 + 4 = 0$

$x^4 - (-4) = 0$

$(D^4+4)y = 0.$ $x^4 - (i2)^2 = 0 \Rightarrow x^4 = (i)^2$

$\Rightarrow \lambda^4 + 4 = 0 \Rightarrow \lambda = \pm 2i, \pm 2i \Rightarrow x^2 = \pm 2i.$

$\Rightarrow x =$

Hence $y = A\cos 2x + B\sin 2x + C\cos 2x + D\sin 2x$

$\Rightarrow y = (A+C)\cos 2x + (B+D)\sin 2x.$ — (1)

$\Rightarrow y' = -2(A+C)\sin 2x + 2(B+D)\cos 2x$ — (2)

$\Rightarrow y'' = -4(A+C)\cos 2x - 4(B+D)\sin 2x$ — (3)

$\Rightarrow y''' = 8(A+C)\sin 2x - 8(B+D)\cos 2x$ — (4)

Applying Initial conditions.

(1) $\Rightarrow A+C = 0 \Rightarrow A = -C.$

(2) $\Rightarrow 2 = 2(B+D) \Rightarrow B+D = 1 \Rightarrow D = 1 - B.$

(3) $\Rightarrow 0 = A$

(4) $\Rightarrow 4 = -8(B+D) \Rightarrow B+D = -\frac{1}{2}.$

$\Rightarrow \cancel{B+1-B = ...}$

$\Rightarrow$

$y = (A\cos x + B\sin x)e^x + (C\cos x + D\sin x)e^{-x}.$

$\Rightarrow y' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) - e^{-x}(C\cos x + D\sin x)$
$+ e^{-x}(-C\sin x + D\cos x)$

$\Rightarrow y' = e^x(A\cos x - A\sin x + B\sin x + B\cos x) + e^{-x}(-C\cos x$
$- C\sin x - D\sin x + D\cos x)$

$\Rightarrow y'' = e^x(A\cos x - A\sin x + B\sin x + B\cos x) + e^x(-A\sin x - A\cos x$
$+ B\cos x - B\sin x) + e^{-x}(C\cos x + C\sin x + D\sin x$
$- D\cos x).$

$$= \left(\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right)$$

$$=$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

$$= \cos\frac{1}{2}(2k\pi + \pi/2) + i\sin$$

$$= \cos(k\pi + \pi/4) + i\sin(k\pi + \pi/4)$$

$k = 0$ or 1

$$=$$

$$\frac{1}{2} = (\cos n\theta - i\sin n\theta)$$

$y(0) = y'(0) = y''(0) = y'''(0) = 1.$

Sol

$$W = \begin{vmatrix} \cosh x & \sinh x & \cos x & \sin x \\ \sinh x & \cosh x & -\sin x & \cos x \\ \cosh x & \sinh x & -\cos x & -\sin x \\ \sinh x & \cosh x & \sin x & -\cos x \end{vmatrix}$$

Q6

~~$\Rightarrow W = \cosh x \sinh x \cos x \sin x$~~

$\Rightarrow W =$

$(D^4-1)y=0 \qquad D^4-1=0 \Rightarrow (D^2+1)(D^2-1)=0$

$\Rightarrow D = 1,1,1,1 \cdot \quad (D^2+1)(D-1)(D+1)=0$

~~$\therefore y = c_1 + c_2 x + c_3 x^2 + c_4 x^3.$~~

$D = 1, -1, \pm i$

$y_h = c_1' e^x + c_2' e^{-x} + c_3 \cos x + c_4 \sin x$

$= \left(\frac{c_1'+c_2'}{2} + \frac{c_1'-c_2'}{2}\right)e^x + \left(\frac{c_1'+c_2'}{2} - \frac{c_1'-c_2'}{2}\right)e^{-x}$

$y = A\cosh x + B\sinh x + C\cos x + D\sin x$ — ①

$\Rightarrow y' = A\sinh x + B\cosh x - C\sin x + D\cos x$ — ②

$\Rightarrow y'' = A\cosh x + B\sinh x - C\cos x - D\sin x$ — ③

$\Rightarrow y''' = A\sinh x + B\cosh x + C\sin x - D\cos x$ — ④

Applying initial condition

① $\Rightarrow 1 = A + C \Rightarrow A = 1 - C \Rightarrow A = 1$

② $\Rightarrow 1 = B + D \Rightarrow B = 1$

③ $\Rightarrow 1 = A - C \Rightarrow 1 = 1 - C - C \Rightarrow -2C = 0 \Rightarrow C = 0.$

④ $\Rightarrow 1 = B - D \Rightarrow D = 0.$

$\therefore y = \cosh x + \sinh x.$

$$x^3y''' + 3x^2y'' = 0 \quad \text{—} (1)$$

$\Rightarrow$ Let $y = x^m$

$\Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

Q7 $\Rightarrow y''' = m(m-1)(m-2)x^{m-3}$

putting in (1)

$$\Rightarrow x^3(m(m-1)(m-2)x^{m-3} + 3x^2 m(m-1)x^{m-2} = 0$$

$\Rightarrow m(m-1)(m-2) + 3m(m-1) = 0$

$\Rightarrow m(m-1)((m-2)+3) = 0$

$\Rightarrow m(m-1)(m+1) = 0$

$\Rightarrow m = 0, 1, 1$

$\therefore y = c_1x^0 + (c_2 + c_3\ln x)x$ — (2)

$\Rightarrow y' = c_2 + c_3\ln x + x c_3(1/x)$

$\Rightarrow y' = c_2 + c_3\ln x + c_3$ — (3)

$\Rightarrow y'' = \frac{c_3}{x}$ — (4)

Applying Initial Conditions

(4) $\Rightarrow 10 = \frac{c_3}{1} \Rightarrow c_3 = 10$

(3) $\Rightarrow -8 = c_2 + c_3 \Rightarrow c_2 = -8 - 10 = -18$

(2) $\Rightarrow 4 = c_1 + c_2 \Rightarrow c_1 = 4 + 18 = 22$

$\therefore y = 22 + (-18 + 10\ln x)x$

(2) $\Rightarrow y = c_1 + c_2x + c_3/x$

(3) $\Rightarrow y' = c_2 - \frac{c_3}{x^2}$

(4) $\Rightarrow y'' = \frac{2c_3}{x^3}$

applying initial conditions.

(4) $\Rightarrow 10 = 2c_3 \Rightarrow c_3 = 5$

(3) $\Rightarrow -8 = c_2 - 5 \Rightarrow c_2 = -3$

(2) $\Rightarrow 4 = c_1 - 3 + 5 \Rightarrow c_1 = 2$

$\therefore y = 2 - 3x + 5/x$

Ans

$$\begin{vmatrix} 0 & 1 & -1/x^2 \\ 0 & 0 & 2/x^3 \end{vmatrix}$$

$$\Rightarrow W = 1\begin{vmatrix} 1 & -1/x^2 \\ 0 & 2/x^3 \end{vmatrix} \Rightarrow W = 2/x^3 \neq 0$$

hence they form the basis

(8) $\cos x, \sin x, \cos 2x, \sin 2x$;

$(D^4+5D^2+4)y=0$; $y(0)=1$, $y'(0)=1$, $y''(0)=-1$

$y'''(0)=-4$

Sol

$$W = \begin{vmatrix} \cos x & \sin x & \cos 2x & \sin 2x \\ -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ -\cos x & -\sin x & -4\cos 2x & -4\sin 2x \\ \sin x & -\cos x & 8\sin 2x & -8\cos 2x \end{vmatrix}$$

$$\Rightarrow W = \begin{vmatrix} 0 & 0 & -3\cos 2x & -3\sin 2x \\ 0 & 0 & 6\sin 2x & -6\cos 2x \\ 0 & -\cos^2 x - \sin^2 x & +8\sin 2x\cos x - 4\cos 2x\sin x & -8\cos 2x\cos x - 4\sin 2x\sin x \\ \sin x & -\cos x & 8\sin 2x & -8\cos 2x \end{vmatrix} \begin{matrix} R_1+R_3 \\ R_2+R_4 \\ R_3\sin x+\cos x\,R_4 \\ \end{matrix}$$

$$\Rightarrow W = \sin x \begin{vmatrix} 0 & -3\cos 2x & -3\sin 2x \\ 0 & 6\sin 2x & -6\cos 2x \\ -1 & 8\sin 2x\cos x - 4\cos 2x\sin x & -8\cos 2x\cos x - 4\sin 2x\sin x \end{vmatrix}$$

$$\Rightarrow W = -\sin x.\begin{vmatrix} -3\cos 2x & -3\sin 2x \\ 6\sin 2x & -6\cos 2x \end{vmatrix}.$$

$$\Rightarrow W = -\sin x(18) = -18\sin \neq 0.$$

hence. they form basis.

$(D^4+5D^2+4)=0.$

$\Rightarrow \lambda^4+5\lambda^2+4=0 \Rightarrow \lambda^4+\lambda^2+4\lambda^2+4=0$

$\Rightarrow \lambda^2(\lambda^2+1)+4(\lambda^2+1)=0$

$\Rightarrow \lambda = \pm i, \pm 2i.$

$\therefore y = A\cos x + B\sin x + C\cos 2x + D\sin 2x.$

$y' = -A\sin x + B\cos x - 2C\sin 2x + 2D\cos 2x.$

$\Rightarrow y'' = -A\cos x - B\sin x - 4C\cos 2x - 4D\sin 2x$

$\Rightarrow y''' = A\sin x - B\cos x + 8C\sin 2x - 8D\cos 2x$

Applying Initial condition.

$1 = A + C$ — ①

$1 = 2B + D$ — ②

$-1 = -A - 4C$ — ③

$-4 = -8B - D$ — ④

①+③ $\Rightarrow 0 = -3C \Rightarrow C = 0 \Rightarrow A = 1.$

②+④ $\Rightarrow -6B = -3 \Rightarrow B = \frac{1}{2} \Rightarrow D = 0$

$\Rightarrow D = \frac{1}{2}$

$\therefore y = \cos x + \frac{1}{2}\sin 2x.$

Ans

(9) $e^{2x}, e^{-2x}, \cos x, \sin x, (D^4 - 3D^2 - 4)y = 0,$