DIFFERENTIAL EQUATIONS

EXERCISE 2.6

Problems solved by;

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Q# 01

GENERAL SOLUTION:

Find a real general Solution.

(2) $x^2y'' - 4xy' + 6y = 0$

The characteristic eq is.

$m^2 + (-4-1)m + 6 = 0$

$\Rightarrow m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0 \Rightarrow m(m-3) - 2(m-3) = 0.$

hence $m_1 = 2$ and $m_2 = 3$

(CASE - I) distinct Roots

Hence the general Solution is.

$y = c_1 x^{m_1} + c_2 x^{m_2} = c_1 x^2 + c_2 x^3.$

Ans

(3) $x^2y'' - 20y = 0$

~~The characteristic eq is.~~

~~$m^2 - 20 = 0 \Rightarrow m = \pm\sqrt{20} \Rightarrow m = \pm 2\sqrt{5}$~~

~~(CASE - I)~~

~~Hence the general solution is given by~~

~~$y = c_1 x^{2\sqrt{5}} + c_2 x^{-2\sqrt{5}}$~~

Ans

The characteristic equation is.

Q3 $m^2+(0-1)m-20=0 \Rightarrow m^2-m-20=0.$

$\therefore m = \dfrac{1 \pm \sqrt{(1)^2-4(-20)}}{2} = \dfrac{1 \pm \sqrt{81}}{2}$

$\Rightarrow m_1 = 5, \quad m_2 = -4.$ (CASE – I)

Hence the G.S is.

$y = c_1 x^5 + c_2 x^{-4}$. Ans

(4) $xy'' + 2y' = 0 \Rightarrow x^2y'' + 2xy' = 0. \quad \text{—— (1)}$

The characteristic eq is.

$m^2 + (2-1)m + 0 = 0 \Rightarrow m^2 + m = 0.$

$\Rightarrow m(m+1) = 0, \Rightarrow m_1 = 0, \; m_2 = -1$ (CASE – I)

Hence the general solution is.

$y = c_1 x^0 + c_2 x^{-1} = c_1 + c_2 x^{-1}$. Ans

Check $y' = -c_2 x^{-2}, \; y'' = 2c_2 x^{-3}$. putting in (1)

$\Rightarrow x^2(2c_2 x^{-3}) + 2x(-c_2 x^{-2}) = 2c_2 x^{-1} - 2c_2 x^{-1} = 0$

hence satisfied.

(5) $10x^2y'' + 46xy' + 32.4y = 0$

$\Rightarrow x^2y'' + 4.6xy' + 3.24y = 0,$

The characteristic eq is.

$m^2 + (4.6-1)m + 3.24 = 0.$

$\Rightarrow m^2 + 3.6m + 3.24 = 0$

$\Rightarrow m = \dfrac{-3.6 \pm \sqrt{(3.6)^2 - 4(3.24)}}{2}$

$\Rightarrow m = \dfrac{-3.6 \pm \sqrt{0}}{2} = \dfrac{-3.6}{2} = -1.8.$ (CASE II) double Roots

Hence the G.S is.

$y = (c_1 + c_2 \ln x)\, x^{-1.8}$

Ans

(6) $x^2y'' - xy' + 2y = 0.$

The characteristic equation is.

$m^2 + (-1-1)m + 2 = 0.$

$\Rightarrow m^2 - 2m + 2 = 0.$ complex Roots

$\Rightarrow m = \dfrac{2 \pm \sqrt{4-4(2)}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$ (CASE – III)

Hence the G.S is.

$y = x^1(A\cos\ln x + B\sin\ln x)$

Ans

(7) $x^2y'' + xy' + y = 0$

The characteristic eq is:

$m^2 + (1-1)m + 1 = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$.

(CASE-III)

Hence the general Solutions.

$y = x^0(A\cos(\ln x) + B\sin(\ln x))$

$\Rightarrow y = A\cos(\ln x) + B\sin(\ln x)$

(8) $(xD^2 + D)y = 0$

$\Rightarrow xy'' + y' = 0$.

$\Rightarrow x^2y'' + xy' = 0$. is the standard form.

and the characteristic eq is:

$m^2 + (1-1)m = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0$. (CASE-II)

Hence the general Solution is.

$y = (c_1 + c_2\ln x)x^0 \Rightarrow y = (c_1 + c_2\ln x)$

Ans

(9) $(4x^2D^2 + 12xD + 3)y = 0$

$\Rightarrow 4x^2y'' + 12xy' + 3 = 0$

$\Rightarrow x^2y'' + 3xy' + 3/4 = 0$ is the standard form.

and the characteristic eq is

$m^2 + (3-1)m + 3/4 = 0$.

$\Rightarrow m^2 + 2m + 3/4 = 0$.

$\Rightarrow m = \frac{-2 \pm \sqrt{4 - 4(3/4)}}{2} = -1 \pm \frac{\sqrt{4-3}}{2} = -1 \pm \frac{1}{2}$.

$\Rightarrow m_1 = -1 + 1/2 = -1/2$.

& $m_2 = -1 - 1/2 = -3/2$.

(CASE -I)

Hence the general Solution is.

$y = c_1x^{-1/2} + c_2x^{-3/2}$ (Ans)

$\Rightarrow x^2y'' - 0.2xy' + 0.36y = 0$ is the standard form and the characteristic eq is

Q12

$m^2 + (-0.2 - 1)m + 0.36 = 0$

$\Rightarrow m^2 - 1.2m + 0.36 = 0$

$\Rightarrow m = \frac{1.2 \pm \sqrt{1.44 - 4(0.36)}}{2} =$

$\Rightarrow m = \frac{1.2}{2} \pm 0 \Rightarrow m = 0.6.$ (CASE - II double roots)

Hence the G.S is

$y = (c_1 + c_2 \ln x)x^{0.6}$ Ans

(10) $(x^2D^2 + 0.7xD - 0.1)y = 0$

$\Rightarrow x^2y'' + 0.7xy' - 0.1y = 0.$

The characteristic equation is

$m^2 + (0.7 - 1)m - 0.1 = 0.$

$\Rightarrow m^2 - 0.3m - 0.1 = 0.$

$\Rightarrow m = \frac{0.3 \pm \sqrt{0.49}}{2} = \frac{0.3 \pm 0.7}{2}.$

$\Rightarrow m_1 = 0.5$ & $m_2 = -0.2.$ CASE - I

Hence the G.S is.

$y = c_1x^{0.5} + c_2x^{-0.2}.$

(11) $(x^2D^2 + 1.25)y = 0.$

$\Rightarrow x^2y'' + 1.25y = 0$

The characteristic eq is.

$m^2 + (0 - 1)m + 1.25 = 0.$

$\Rightarrow m^2 - m + 1.25 = 0$

$\Rightarrow m = \frac{+1 \pm \sqrt{1 - 4(1.25)}}{2} = \frac{+1 \pm \sqrt{-4}}{2}$

$\Rightarrow m = 0.5 \pm i$ (CASE - III)

Hence the general solution is.

$y = x^{0.5}(A\cos(\ln x) + B\sin(\ln x))$

Ans

$\Rightarrow x^2y'' + 7xy' + 9y = 0.$

Q13 The characteristic eq is-

$m^2 + 6m + 9 = 0$

$\Rightarrow m = \dfrac{-6 \pm \sqrt{36 - 9(4)}}{2} \Rightarrow m = -3 \cdot$ (CASE - II)

Hence the g.s is

$y = (c_1 + c_2 \ln x)x^{-3}$ Ans

Questions are Remaining in the last. . .

EXERCISE 2.7

① $e^{\lambda_1 x}, e^{\lambda_2 x}$

$$W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = \lambda_2 e^{\lambda_2 x} e^{\lambda_1 x} - e^{\lambda_2 x} \cdot \lambda_1 e^{\lambda_1 x}.$$

$\Rightarrow W = e^{\lambda_1 x + \lambda_2 x}(\lambda_2 - \lambda_1)$

$\Rightarrow W = e^{x(\lambda_1 + \lambda_2)}(\lambda_2 - \lambda_1) = (\lambda_2 - \lambda_1)e^{x(\lambda_1 + \lambda_2)}$

hence $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent iff $\lambda_2 \neq \lambda_1$.

② $1, e^x$.

$$W = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x \neq 0$$

hence 1 and e^x are L.I

③ $e^{-ax/2}\cos 3x, \; e^{-ax/2}\sin 3x.$

$$W = \begin{vmatrix} e^{-ax/2}\cos 3x & e^{-ax/2}\sin 3x \\ -\frac{a}{2}e^{-ax/2}\cos 3x - 3e^{-ax/2}\sin 3x & -\frac{a}{2}e^{-\frac{ax}{2}}\sin 3x + 3e^{-\frac{ax}{2}}\cos 3x \end{vmatrix}$$

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$\Rightarrow W = e^{-\frac{ax}{2}} \cdot \cos 3x \left(-\frac{a}{2} e^{-\frac{ax}{2}} \sin 3x + 3 e^{-\frac{ax}{2}} \cos 3x\right)$

$\quad - e^{-ax/2} \sin 3x \left(-\frac{a}{2} e^{-\frac{ax}{2}} \cos 3x - 3 e^{-\frac{ax}{2}} \sin 3x\right)$

$\Rightarrow W = \cancel{-\frac{a}{2} e^{-\frac{ax}{2}} \cos 3x \sin 3x} + 3 e^{-ax} \cos^2 3x + \cancel{\frac{a}{2} e^{-\frac{ax}{2}} \cos 3x \sin 3x}$

$\quad + 3 e^{-\frac{ax}{2}} \sin^2 3x .$

$\Rightarrow W = 3 e^{-ax} (\cos^2 3x + \sin^2 3x) = 3 e^{-ax} .$

④ x^{m_1}, x^{m_2}

$\Rightarrow W = \begin{vmatrix} x^{m_1} & x^{m_2} \\ m_1 x^{m_1 - 1} & m_2 x^{m_2 - 1} \end{vmatrix} = x^{m_1}(m_2 x^{m_2 - 1}) - x^{m_2}(m_1 x^{m_1 - 1})$

$= m_2 x^{m_1 + m_2 - 1} - m_1 x^{m_1 + m_2 - 1}$

$\Rightarrow x^{m_1 + m_2 - 1}(m_2 - m_1)$

⑤ $x^4, x^4 \ln x$

$\Rightarrow W = \begin{vmatrix} x^4 & x^4 \ln x \\ 4x^3 & 4x^3 \ln x + x^3 \end{vmatrix} = x^4(4x^3 \ln x + x^3) - 4x^3(x^4 \ln x)$

$= \cancel{4x^7 \ln x} + x^7 - \cancel{4x^7 \ln x}$

$= x^7 .$

Hence y_1 and y_2 will be L.I iff $x \neq 0$.

⑥ $e^{\lambda x}, x e^{\lambda x}$

$\Rightarrow W = \begin{vmatrix} e^{\lambda x} & x e^{\lambda x} \\ \lambda e^{\lambda x} & \lambda x e^{\lambda x} + e^{\lambda x} \end{vmatrix}$

$\Rightarrow W = e^{\lambda x}(\lambda x e^{\lambda x} + e^{\lambda x}) - x e^{\lambda x}(\lambda e^{\lambda x}) .$

$\Rightarrow W = \lambda x e^{2\lambda x} + e^{2\lambda x} - \lambda x e^{2\lambda x}$

$\Rightarrow W = e^{2\lambda x} .$

Hence the two sol will be L.I for any Real x.

$$W = \begin{vmatrix} x^{\mu}\cos(2\ln x) & x^{\mu}\sin(2\ln x) \\ \mu x^{\mu-1}\cos(2\ln x) + x^{\mu}\left(-\frac{2}{x}\sin(2\ln x)\right) & \mu x^{\mu-1}\sin 2\ln x + \frac{2}{x}x^{\mu}\cos(2\ln x) \end{vmatrix}$$

$$\Rightarrow W = x^{\mu}\cos(2\ln x)\left(\mu x^{\mu-1}\sin 2\ln x + 2x^{\mu-1}\cos(2\ln x)\right)$$
$$- x^{\mu}\sin(2\ln x)\left(\mu x^{\mu-1}\cos(2\ln x) - 2x^{\mu-1}\sin 2\ln x\right)$$

$\Rightarrow W = 2x^{2\mu-1}$. Ans

(8) $e^{-x}\cos\omega x$, $e^{-x}\sin\omega x$.

$$W = \begin{vmatrix} e^{-x}\cos\omega x & e^{-x}\sin\omega x \\ -e^{-x}\cos\omega x - \omega e^{-x}\sin\omega x & -e^{-x}\sin\omega x + \omega e^{-x}\cos\omega x \end{vmatrix}$$

$$\Rightarrow W = e^{-x}\cos\omega x\left(-e^{-x}\sin\omega x + \omega e^{-x}\cos\omega x\right)$$
$$- e^{-x}\sin\omega x\left(-e^{-x}\cos\omega x - \omega e^{-x}\sin\omega x\right)$$

$\Rightarrow W = \omega e^{-2x}$. Ans

EQUATIONS FOR GIVEN BASES

Find a 2nd order homogeneous linear differential eq for which the given functions are solutions. Find wronskian and use it to find the L.I by Theorem - 2

(9) e^{3x}, xe^{3x}

REMEMBER.

That the standard 2nd order L.D.E is.

$y'' + ay' + by = 0$ —— (1)

in this a and b can be found as ·

& $b = \lambda_1 \lambda_2$

And for Cauchy Equation

add +1 with a because in characteristic equation we write

$$m^2 + (a-1)m + b = 0.$$

So Here
we have $y_1 = e^{3x}$ & $y_2 = xe^{3x}$.

$\lambda_1 = 3$, $\lambda_2 = 3$.

$\therefore a = -(\lambda_1 + \lambda_2) = -(3+3) = -6.$

and $b = \lambda_1 \lambda_2 = 3 \times 3 = 9$.

hence the general second order linear differential is.

$$y'' - 6y' + 9y = 0$$

Check : The characteristic eq is.

$$\lambda^2 - 6\lambda + 9 = 0.$$

$\Rightarrow \lambda^2 - 3\lambda - 3\lambda + 9 = 0$

$\Rightarrow \lambda(\lambda - 3) - 3(\lambda - 3) = 0$

$\Rightarrow \lambda = 3$ double roots (CASE – II)

$\therefore y_1 = e^{3x}$ and $y_2 = xe^{3x}$.

REMAINING QUESTIONS ARE IN THE LAST (i.e) PAGES AFTER The NEXT 2.