DIFFERENTIAL EQUATIONS

EXERCISE 2.2,2.3

Problems solved by;

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(*) General Solutions: Find general sol. Check your answer by substitution.

① $4y'' + 4y' - 3y = 0$ —— ①

Let $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ putting in ①

$\Rightarrow (4\lambda^2 + 4\lambda - 3)e^{\lambda x} = 0$

$\Rightarrow 4\lambda^2 + 4\lambda - 3 = 0$ (characteristic equation)

$\Rightarrow 4\lambda^2 + 6\lambda - 2\lambda - 3 = 0$

$\Rightarrow 2\lambda(2\lambda + 3) - 1(2\lambda + 3) = 0 \Rightarrow 2\lambda - 1 = 0$ & $2\lambda + 3 = 0$

$\Rightarrow \lambda_1 = 1/2$ & $\lambda_2 = -3/2$.

∴ the solutions are:

$y_1 = e^{x/2}$ & $y_2 = e^{-3x/2}$ and the general solution is

$y = c_1 y_1 + c_2 y_2$ (∵ y_1 and y_2 are linearly independent)

$\Rightarrow y = c_1 e^{x/2} + c_2 e^{-3x/2}$

Check $y = c_1 e^{x/2} + c_2 e^{-3x/2}$.

$$\left.\begin{aligned} &\Rightarrow y' = \frac{c_1}{2} e^{x/2} - \frac{3c_2}{2} e^{-3x/2} \\ &\Rightarrow y'' = \frac{c_1}{4} e^{x/2} + \frac{9c_2}{4} e^{-3x/2} \end{aligned}\right\} \text{putting in ①}$$

$\Rightarrow \cancel{c_1 e^{x/2}} + \cancel{9c_2 e^{-3x/2}} + \cancel{2c_1 e^{x/2}} - \cancel{6c_2/2\, e^{-3x/2}} - \cancel{3c_1 e^{x/2}} - \cancel{3c_2 e^{-3x/2}}$

$= 0$

Hence our solution is correct.

② $y'' + 3.2y' + 2.56y = 0$

$\Rightarrow 10y'' + 32y' + 25.6y = 0 \Rightarrow 100y'' + 320y' + 256y = 0$

The characteristic equation is $100\lambda^2 + 320\lambda + 256 = 0$.

$$\lambda = \frac{-320 \pm \sqrt{(320)^2 - 4(100)(256)}}{2(100)}$$

$$\Rightarrow \lambda = \frac{-320 \pm 0}{2(100)}$$

$\Rightarrow \lambda = \lambda_1 = \lambda_2 = -1.6$

Hence the general solution is.

$y = c_1 y_1 + c_2 y_2$

where $y_2 = u y_1$

$u = \int v\,dx = \int \frac{e^{-\int a\,dx}}{y_1^2}\,dx$, here $a = 3.2$

$\therefore\ u = \int \frac{e^{-3.2x}}{(e^{-1.6x})^2}\,dx = \int dx = x$

$\therefore\ y_2 = x e^{-1.6x}$

So the general solution is

$y = (c_1 + c_2 x) e^{-1.6x}$

Check

$y' = (c_1 + c_2 x)(-1.6 e^{-1.6x}) + c_2 e^{-1.6x}$

$= -1.6 e^{-1.6x} - 1.6 c_2 x e^{-1.6x} + c_2 e^{-1.6x}$

Q#3 $2y'' - 9y' = 0$ —(1)

Sol the characteristic/auxiliary equation is.

$2\lambda^2 - 9\lambda = 0$

$\Rightarrow \lambda(2\lambda - 9) = 0$

$\Rightarrow \lambda = 0,\ \lambda = 9/2.$ (Case-I)

$\therefore$ the solution is.

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{0} + c_2 e^{(9/2)x}$

$\therefore\ y = c_1 + c_2 e^{9x/2}$. Ans

Check $y' = \frac{9}{2} c_2 e^{9x/2}$, $y'' = \frac{81}{4} c_2 e^{9x/2}$. putting in (1)

$\Rightarrow 2\left(\frac{81}{4} c_2 e^{9x/2}\right) - 9\left(\frac{9}{2} e^{9x/2}\right) = 0$

satisfied

(4) $y'' - 8y = 0$

The characteristic eq is

$\lambda^2 - 8 = 0 \Rightarrow \lambda = \pm 2\sqrt{2}$.

$\therefore y = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x}$ is the general sol.

Check $y' = 2\sqrt{2}c_1 e^{2\sqrt{2}x} - 2\sqrt{2}c_2 e^{-2\sqrt{2}x}$

$\Rightarrow y'' = 8c_1 e^{2\sqrt{2}x} + 8c_2 e^{-2\sqrt{2}x}$. putting in (1)

(1) $\Rightarrow 8c_1 e^{2\sqrt{2}x} + 8c_2 e^{-2\sqrt{2}x} - 8(c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x}) = 0$

satisfied

(5) $y'' + 9y' + 20y = 0$

The characteristic equation is

$\lambda^2 + 9\lambda + 20 = 0$

$\Rightarrow \lambda^2 + 5\lambda + 4\lambda + 20 = 0$

$\Rightarrow \lambda(\lambda+5) + 4(\lambda+5) = 0 \Rightarrow \lambda = -5, \lambda = -4$

(CASE – I)

Hence the sol is

$y = c_1 e^{-5x} + c_2 e^{-4x}$.

Ans

(6) $16y'' - \pi^2 y = 0$

The characteristic equation is

$16\lambda^2 - \pi^2 = 0 \Rightarrow \lambda^2 = \frac{\pi^2}{16} \Rightarrow \lambda = \pm \pi/4$ (CASE – I

Hence the general sol is

$y = c_1 e^{\pi x/4} + c_2 e^{-\pi x/4}$.

Ans

(7) $9y'' - 30y' + 25y = 0$

The characteristic equation is

$9\lambda^2 - 30\lambda + 25 = 0 \Rightarrow 9\lambda^2 - 15\lambda - 15\lambda + 25 = 0$

$\Rightarrow 3\lambda(3\lambda - 5) - 5(3\lambda - 5) = 0$

$\Rightarrow \lambda = 5/3,\ \lambda = 5/3$ (CASE – II)

∴ The general solution is

$y = c_1 e^{+5x/3} + c_2 x e^{+5x/3}$

(8) $10y'' + 6y' - 4y = 0$ —— (1)

The characteristic equation is

$10\lambda^2 + 6\lambda - 4 = 0$

$\lambda = \dfrac{-6 \pm \sqrt{36 - (-160)}}{2(10)} = \dfrac{-6 \pm 14}{20}$

$\Rightarrow \lambda_1 = \cancel{8}^{4^2}/\cancel{20}_{10\,5} = 2/5,\ \lambda_2 = -2$ (CASE – I)

Hence the general sol is

$y = c_1 e^{2x/5} + c_2 e^{-x}$

Ans

Check

$y' = \frac{2}{5} c_1 e^{2x/5} - c_2 e^{-x} \Rightarrow y'' = \frac{4}{25} c_1 e^{2x/5} + c_2 e^{-x}$

putting in (1)

$\Rightarrow 10\left(\frac{4}{25} c_1 e^{2x/5} + c_2 e^{-x}\right) + 6\left(\frac{2}{5} c_1 e^{2x/5} - c_2 e^{-x}\right)$

$- 4\left(c_1 e^{2x/5} + c_2 e^{-x}\right) = 0$

satisfied

(9) $y'' + 2ky' + k^2 y = 0$

The characteristic eq is

$\lambda^2 + 2k\lambda + k^2 = 0$

$\Rightarrow \lambda^2 + k\lambda + k\lambda + k^2 = 0 \Rightarrow \lambda(\lambda + k) + k(\lambda + k)$

$\Rightarrow \lambda = -k$ (CASE – II)

$\Rightarrow$ Solve the following Initial value Problems.

(10) $y''+y'-6y=0$ —(a), $y(0)=10$, $y'(0)=0$

The characteristic eq is:

$\lambda^2+\lambda-6=0 \Rightarrow \lambda^2+3\lambda-2\lambda-6=0$.

$\Rightarrow \lambda(\lambda+3)-2(\lambda+3)=0$

$\Rightarrow \lambda=2, \lambda=-3$. (CASE – I)

Hence the general solution is:

$y=c_1e^{2x}+c_2e^{-3x}$ —(1) $\Rightarrow y'=2c_1e^{2x}-3c_2e^{-3x}$ —(2)

From initial values we have.

$10=c_1+c_2$ —(3) & $0=2c_1-3c_2$ —(4)

Xing (3) with "+3" and adding with (4)

$\Rightarrow 30=5c_1 \Rightarrow c_1=6$. putting in (3) or (4)

$\Rightarrow c_2=4$ putting c_1 and c_2 in (1)

$\Rightarrow y=6e^{2x}+4e^{-3x}$ we got the particular sol

Check $y'=12e^{2x}-12e^{-3x}$, $y''=24e^{2x}+36e^{-3x}$ putting in (a)

$\Rightarrow \cancel{12e^{2x}}-\cancel{12e^{-3x}}+\cancel{24e^{2x}}+\cancel{36e^{-3x}}-\cancel{36e^{2x}}-\cancel{24e^{-3x}}=0$

satisfied

(11) $y''+4y'+4y=0$, $y(0)=1$, $y'(0)=1$.

The characteristic equation is.

$\lambda^2+4\lambda+4=0 \Rightarrow \lambda^2+2\lambda+2\lambda+4=0$.

$\Rightarrow \lambda(\lambda+2)+2(\lambda+2)=0$

$\Rightarrow \lambda=-2$ (CASE I).

Hence the general sol is -

$y=(c_1+c_2x)e^{-2x}$ —(1)

$\Rightarrow y'=-2c_1e^{-2x}+c_2e^{-2x}-2xc_2e^{-2x}$

From Initial values we have.

$1=(c_1+0) \Rightarrow c_1=1$

& $1=-2c_1+c_2 \Rightarrow c_2=3$ putting in (1) to

(13) $y'' - y = 0$, $y(0) = 3$, $y'(0) = -3$

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$ CASE I

$\therefore$ the general sol is

$y = c_1 e^{x} + c_2 e^{-x}$. —— (1)

Also $y' = c_1 e^{x} - c_2 e^{-x}$. —— (2)

From initial value conditions we have

(1) $\Rightarrow 3 = c_1 + c_2$ —— (3)

(2) $\Rightarrow -3 = c_1 - c_2$ —— (4)

adding (3) and (4)

$\Rightarrow c_1 = 0$ and Thus $c_2 = 3$ putting in (1)

$\Rightarrow y = 3e^{-x}$ is the particular sol.

Check

$y' = -3e^{-x}$, $y'' = 3e^{-x}$

$\therefore y'' - y = 0$. satisfies

(14) $4y'' - 25y = 0$, $y(0) = 0$, $y'(0) = -5$

The characteristic eq is

$\Rightarrow 4\lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5/2$. CASE I

$\therefore y = c_1 e^{5x/2} + c_2 e^{-5x/2}$. —— (1)

Also $y' = \frac{5}{2} c_1 e^{5x/2} - 5/2\, c_2 e^{-5x/2}$ —— (2)

From initial conditions we have from

Also from (1) we get

(2) $\Rightarrow -5 = \frac{5}{2} c_1 - 5/2\, c_2$; $c_1 = -c_2$. so.

(2) $\Rightarrow -5 = 5/2\, c_1 + 5/2\, c_1 \Rightarrow$

$\Rightarrow 2(5/2) c_1 = -5 \Rightarrow c_1 = -1 \Rightarrow c_2 = 1$. putting in (1)

$\Rightarrow y = -e^{5x/2} + e^{5x/2} = 0$

is the particular sol

$y'' = 0$

Hence $4y'' - 25y = 0 - 0 = 0$

satisfied

(16) $y'' - k^2 y = 0 \quad , k \neq 0 , \quad y(0) = 1, \; y'(0) = 1$

the characteristic eq is

$\lambda^2 - k^2 = 0 \Rightarrow \lambda = \pm k$ (CASE I)

the general sol is

$y = c_1 e^{kx} + c_2 e^{-kx}$ —— (1)

Also $y'' = kc_1 e^{kx} - kc_2 e^{-kx}$ —— (2)

putting Initial conditions

$(1) \Rightarrow 1 = c_1 + c_2$ → (3), $(2) \Rightarrow 1 = kc_1 - kc_2$ → (4)

comparing (3) and (4)

$\Rightarrow c_1 + c_2 = kc_1 - kc_2 \Rightarrow c_1 - kc_1 = -c_2 - kc_2$

$\Rightarrow (1-k)c_1 = (-1-k)c_2$ —— (5)

Multiplying (3) with k and adding with (4)

$\Rightarrow k = kc_1 + kc_2$

$\underline{1 = kc_1 - kc_2}$

$\Rightarrow k + 1 = 2kc_1 \quad \Rightarrow c_1 = \frac{k+1}{2k}$ putting in (5)

$\Rightarrow \frac{(1-k)(1+k)}{2k} = -(1+k)c_2$

$\Rightarrow c_2 = \frac{k-1}{2k}$ putting in (1)

$\Rightarrow y = \frac{k+1}{2k} e^{kx} + \frac{k-1}{2k} e^{-kx}$

check

$y' = \frac{k(k+1)e^{kx}}{2k} - \frac{k(k-1)e^{-kx}}{2k}$

$\Rightarrow y'' = \frac{k(k+1)}{2} e^{kx} + \frac{k(k-1)}{2} e^{-kx}$ putting in Q.

$\Rightarrow \frac{k(k+1)}{2} e^{kx} + \frac{k(k-1)}{2} e^{-kx} - k^2\left(\frac{k+1}{2k} e^{kx} + \frac{k-1}{2k} e^{-kx}\right)$

$= 0$ hence satisfied

Q17 The characteristic eq is:

$4\lambda^2-4\lambda-3=0 \Rightarrow 4\lambda^2-6\lambda+2\lambda-3=0$

$\Rightarrow 2\lambda(2\lambda-3)+1(2\lambda-3)=0$

$\Rightarrow \lambda = 3/2,\ \lambda = -1/2$ CASE-I

Hence the general sol is:

$y = c_1 e^{3x/2} + c_2 e^{-x/2}$ ——①

also $y' = \frac{3}{2}c_1 e^{3x/2} - \frac{1}{2}c_2 e^{-x/2}$ ——②

putting initial conditions in ① and ②

① $\Rightarrow e = 1 c_1 e^{-3} + 1 c_2 e^{1}$ ——③

② $\Rightarrow -\frac{e}{2} = \frac{3}{2}c_1 e^{-3} - \frac{1}{2}c_2 e^{1}$ ——④

$\Rightarrow -e = 3c_1 e^{-3} - e c_2$ ——⑤

adding ③ and ⑤

$\Rightarrow 0 = 4c_1 e^{-3} - 0 \Rightarrow c_1 = 0$. putting in ③

$\Rightarrow c_2 = 1$ putting $c_1=0$ & $c_2=1$ in ①

$\Rightarrow y = e^{-x/2}$. Ans

LINEAR INDEPENDENCE

Are the following functions linearly independent or dependent on the given intervals.

⑱ e^{-x}, e^{x}, any interval.

Since $\frac{e^{-x}}{e^{x}} = e^{-2x} \neq$ const. for any x.

hence the functions are linearly Independent.

or det $k_1 e^{-x} + k_2 e^{x} = 0$.

This could only be zero where $k_1=0, k_2=0$

because e^{-x} and e^{x} are not zero for any real x

If $y_1(x), y_2(x), \cdots, y_m(x)$ are m functions of an independent variable x and $c_1, c_2, \cdots, c_m$ are constants, then

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_m y_m(x).$$

is called a linear combination of $y_1(x), y_2(x), \cdots, y_m(x)$.

As in vector spaces, the m functions $y_1, y_2, y_3, \cdots, y_m$ are called linearly dependent iff (iff means if and only if) there exists constants $c_1, c_2, \cdots, c_m$, at least one of which is non-zero, such that

$$c_1y_1 + c_2y_2 + \cdots + c_m y_m = 0.$$

The functions $y_1, y_2, y_3, \cdots, y_m$ are called linearly Independent iff they are no linearly dependent, i.e. iff.

$$c_1y_1 + c_2y_2 + \cdots + c_m y_m = 0.$$

implies $c_1 = c_2 = \cdots = c_m = 0$.

Remember

Every homogeneous linear nth-order differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

has n linearly independent solutions, $y_1, y_2, \cdots, y_n$

For a homogeneous differential equation with constant co-efficients.

$$y'' + ay' + by = 0 \quad \text{——} \ (1)$$

The characteristic equation is.

$$\lambda^2 + a\lambda + b = 0 \quad \text{——} \ (2)$$

whose roots are.

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \quad \text{——} \ (3)$$

Complex Roots (Case-III)

If in (3) $a^2 - 4b < 0$ we get the two complex roots.

Example

$y'' + y = 0$. whose characteristic eq is.

$$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1} = \pm i.$$

Hence we get the two solutions as.

$$y_1 = c_1 e^{ix}, \quad y_2 = c_2 e^{-ix}.$$

And the general solution

$$y = c_1 e^{ix} + c_2 e^{-ix} \quad \text{——} \ (4)$$

We can also write.

$$y'' + y = 0 \quad \text{as} \quad y'' = -y.$$

Hence we want the sol ~~that~~ is any function which come back under two differentiation times a minus sign.

As

$$(\cos x)'' = -\cos x \quad \text{and} \quad (\sin x)'' = -\sin x$$

Thus the general solution is

$$y = A\cos x + B\sin x \quad \text{——} \ (5)$$

Now let us consider (4). We know that by Euler formula.

(a) $e^{ix} = \cos x + i\sin x.$

(b) $e^{-ix} = \cos x - i\sin x.$

By adding (a) and (b) we get.

and by subtracting (a) and (b) we get

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad \text{—— 6(b)}$$

Since e^{ix} and e^{-ix} are solutions of (1), so as $\cos x$ and $\sin x$ by direct verification.

COMPLEX EXPONENTIAL FUNCTION

We define the complex Exponential Function e^z of a complex number $z = s + it$. The definition in terms of real functions, e^s, $\cos t$ and $\sin t$ is.

$$e^z = e^{s+it} = e^s \cdot e^{it} = e^s(\cos t + i\sin t) \quad \text{—— (7)}$$

For real $z = s$, the function e^z becomes the real exponential function e^x (because $\cos 0 = 1$ and $\sin 0 = 0$) It can be shown that $e^{z_1 + z_2} = e^{z_1} e^{z_2}$, just as in real.

The Maclaurin series of e^x with $x = it$ gives

As the Maclaurin Series is given by

$$\sum_{k=0}^{\infty} \frac{f^k(0)x^k}{k!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^n(0)x^n}{n!} + \cdots$$

for $f(x) = e^{it}$. The Maclaurin Series is

$$e^{it} = e^{(0)} + e^{(0)}it + \frac{e^{(0)}(it)^2}{2!} + \frac{e^{(0)}(it)^3}{3!} + \frac{e^{(0)}(it)^4}{4!} + \cdots$$

$$= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$

$$= 1 + it - \frac{1t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + it - \frac{it^3}{3!} + \frac{it^5}{5!} + \cdots$$

$$\Rightarrow e^{it} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + it - \frac{it^3}{3!} + \frac{it^5}{5!} + \cdots$$

$$\Rightarrow e^{it} = \cos t + i\sin t.$$

In case III, the radicand.

a^2-4b is negative, hence to make it positive we pull out -1 under the root and use $\sqrt{-1} = i$ as.

$$\lambda_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2-4b} = -\frac{1}{2}a + \frac{1}{2}\sqrt{-(4b-a^2)}$$

$$= -\frac{1}{2}a + \frac{1}{2}\sqrt{-1}\sqrt{4b-a^2} = -\frac{1}{2}a + \frac{1}{2}i\sqrt{4b-a^2}$$

$$= -\frac{1}{2}a + \frac{2}{2}i\sqrt{b-a^2/4}$$

$$\Rightarrow \lambda_1 = -\frac{1}{2}a + i\sqrt{b-\frac{1}{4}a^2} \text{ — (1) and similarly}$$

$$\lambda_2 = -\frac{1}{2}a - i\sqrt{b-\frac{1}{4}a^2} \text{ — (2)}$$

putting $\sqrt{b-\frac{1}{4}a^2} = \omega$ in (1) and (2)

$$\Rightarrow \lambda_1 = -\frac{1}{2}a + i\omega \;\; \& \;\; \lambda_2 = -\frac{1}{2}a - i\omega \text{ — (8)}$$

Using this and applying 7, Our result is

$$e^{\lambda_1 x} = e^{-\frac{a}{2}x + i\omega x} = e^{-\frac{a}{2}x} e^{i\omega x} = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x)$$

$$\& \; e^{\lambda_2 x} = e^{-\frac{a}{2}x - i\omega x} = e^{-\frac{a}{2}x} e^{-i\omega x} = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x)$$

adding both and dividing by (2) we get y_1.

$$y_1 = \frac{1}{2}e^{-ax/2}(\cos\omega x + i\sin\omega x + \cos\omega x - i\sin\omega x)$$

$$\Rightarrow y_1 = \frac{1}{2}e^{-ax/2}(2\cos\omega x)$$

$$\Rightarrow y_1 = e^{-ax/2}\cos\omega \text{ — (9)}$$

subtracting both and dividing by $2i$, we get y_2.

$$\& \; y_2 = \frac{1}{2i}e^{-ax/2}(\cos\omega x + i\sin\omega x - \cos\omega x + i\sin\omega x)$$

$$\Rightarrow y_2 = \frac{1}{2i}e^{-ax/2}(2i\sin\omega x)$$

$$\Rightarrow y_2 = e^{-ax/2}\sin\omega x \text{ — (10)}$$

These are the solutions of (1), as follows by differentiation and substitution, They form a basis because they are linearly independent.

constant because $\omega \neq 0$ (why?) $\Rightarrow$ Because here we have considered $\omega = \sqrt{b - \frac{1}{4}a^2}$ which is positive. If it is zero then the case is of the double roots, whereas here we are considering complex roots. As y_1 and y_2 are not proportional the corresponding general sol is.

$$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x) \text{ ——— } (11)$$

EXAMPLE

Solve

$$y'' + 0.2y' + 4.01y = 0 \quad , \quad y(0) = 0, \; y'(0) = 2.$$

Sol The characteristic equation is given by,

$$\lambda^2 + 0.2\lambda + 4.01 = 0$$

It has the roots.

$$\lambda = -0.2 \pm i\sqrt{4.01 - \tfrac{1}{4}(0.2)^2} \quad \text{by (B)}$$

$$\Rightarrow \lambda = -0.2 \pm 2i \quad , \text{Hence } \omega = 2.$$

and thus the general sol is.

$$y = e^{-0.2x/2}(A\cos 2x + B\sin 2x).$$

$$\Rightarrow y = e^{-0.1x}(A\cos 2x + B\sin 2x) \text{ ——— } (1)$$

Also $y' = -0.1e^{-0.1x}A\cos 2x + e^{-0.1x}(-2A\sin 2x) + (-0.1e^{-0.1x}B\sin 2x + e^{-0.1x}2B\cos 2x)$

~~Applying~~

$$\Rightarrow y' = -(0.1e^{-0.1x})[A\cos 2x + B\sin 2x] + e^{-0.1x}(2B\cos 2x - 2A\sin 2x) \text{ ——— } (2)$$

Applying initial conditions.

$$(1) \Rightarrow 0 = e^{0}(A\cos 0 + B\sin 0) = A \Rightarrow A = 0.$$

$$(2) \Rightarrow 2 = -0.1e^{0}[A\cos 0 + B\sin 0] + e^{0}(2B\cos 0 - 2A\sin 0)$$

$$\Rightarrow -2 = 0.1A - 2B \Rightarrow 2B = +2 + 0.1A.$$

$$\Rightarrow 2B = 2 + 0 \Rightarrow B = 1 \quad \text{putting in (1)}$$

SUMMARY OF CASES I - II & III

Case.	Roots of (2)	Basis of (1)	General Sol.
I	Distinct real. λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real Double roots. $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$.
III	complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$ $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

BOUNDARY VALUE PROBLEMS

Applications sometimes Also leads to the condition of type

$$y(P_1) = k_1 \; , \; y(P_2) = k_2$$

These are known as boundary conditions, since they refer to the endpoints P_1, P_2 of an interval I on which the eq (1) is considered

PROBLEM SET 2.3

Conversion to Real Form: verify that the given function is a solution and derive the corresponding real general solution.

① $y = c_1 e^{(1+i)x} + c_2 e^{(1-i)x}$, $y'' - 2y' + 2y = 0$

$$y' = c_1 e^{(1+i)x} + c_2 e^{(1-i)x}.$$

$$y'' = c_1 e^{(1+i)x} + c_2 e^{(1-i)x}.$$

putting in.

$$y'' - 2y' + 2y = 0$$

$$\Rightarrow c_1 e^{(1+i)x} + c_2 e^{(1+i)x} - 2c_1 e^{(1+i)x} - 2c_2 e^{(1-i)x} +$$

Not Understood.

corresponds to case I, case II, or case III, Find a real general solution (show each step of your derivation)

(5) $25y'' + 40y' + 16y = 0$

The characteristic eq is

$25\lambda^2 + 40\lambda + 16 = 0$

$\Rightarrow 25\lambda^2 + 20\lambda + 20\lambda + 16 = 0 \Rightarrow 5\lambda(5\lambda + 4) + 4(5\lambda + 4) =$

$\Rightarrow \lambda_1 = -4/5,\ \lambda_2 = -4/5$ **CASE-II**

$\therefore$ The general solution is

$y = (c_1 + c_2 x)e^{-ax/2} = (c_1 + c_2 x)e^{\lambda x}$

$\Rightarrow y = (c_1 + c_2 x)e^{-0.8x}$ $\left(\because -a = \frac{-40}{25} \therefore -4/5 = \lambda\right)$ Ans

(6) $y'' + y' - 12y = 0$

The characteristic eq is

$\lambda^2 + \lambda - 12 = 0$

$\Rightarrow \lambda^2 + 4\lambda - 3\lambda - 12 = 0 \Rightarrow \lambda(\lambda + 4) - 3(\lambda + 4) = 0$.

$\Rightarrow \lambda = 3,\ \lambda = -4$. CASE I

Hence the general solution is

$y = c_1 e^{3x} + c_2 e^{-4x}$ Ans

(7) $16y'' - 8y' + 5y = 0$

The characteristic eq is

$16\lambda^2 - 8\lambda + 5 = 0$.

$\lambda = \frac{8 \pm \sqrt{64 - 4(16)(5)}}{32} = \frac{8 \pm \sqrt{-256}}{32}$

$\Rightarrow \lambda = \frac{1}{4} \pm \frac{16i}{32}$ (CASE III)

Hence the general solution is

$y = e^{x/4}(A\cos(1/2)x + B\sin(1/2)x)$ Ans

The characteristic eq is.

$\lambda^2 + 4\lambda + 4 + \omega^2 = 0.$

$\therefore \lambda = \dfrac{-4 \pm \sqrt{16 - 4(4+\omega^2)}}{2}$

$\Rightarrow \lambda = -2 \pm \dfrac{\sqrt{16-16-4\omega^2}}{2}$

$\Rightarrow \lambda = -2 \pm \dfrac{2i\sqrt{\omega^2}}{2}$

$\Rightarrow \lambda = -2 \pm i\omega.$ (CASE-III)

Hence the general solution is.

$y = e^{-2x}(A\cos\omega x + B\sin\omega x)$ Ans

(9) $y'' - 9\pi^2 y = 0$

The characteristic eq is.

$\lambda^2 - 9\pi^2 = 0$

$\Rightarrow \lambda^2 = 9\pi^2 \Rightarrow \lambda = \pm 3\pi.$ (CASE I)

Hence the general sol is.

$y = c_1 e^{3\pi x} + c_2 e^{-3\pi x}$ Ans

(10) $y'' - 2\sqrt{2}y' + 2.5y = 0$

The characteristic eq is.

$\lambda^2 - 2\sqrt{2}\lambda + 2.5 = 0.$

$\Rightarrow \lambda = \dfrac{2\sqrt{2}}{2} \pm \dfrac{\sqrt{8 - 4(2.5)}}{2}$

$\Rightarrow \lambda = \sqrt{2} \pm \dfrac{\sqrt{-2}}{2} = \sqrt{2} \pm \dfrac{i}{\sqrt{2}}.$ (CASE III)

Hence the general sol. is

$y = e^{\sqrt{2}x}\left(A\cos\dfrac{x}{\sqrt{2}} + B\sin\dfrac{x}{\sqrt{2}}\right)$

Ans

The characteristic eq is.

$\lambda^2 - 2\sqrt{2}\lambda + 2 = 0$.

$\Rightarrow \lambda = \frac{2\sqrt{2}}{2} \pm \sqrt{\frac{8-8}{2}} \Rightarrow \lambda = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$ (CASE-II)

Hence the general solution is.

$y = (c_1 + c_2 x)e^{\sqrt{2}x}$

Ans

(12) $y'' + 2ky' + (k^2 + k^{-2})y = 0$

The characteristic eq is

$\lambda^2 + 2k\lambda + k^2 + k^{-2} = 0$.

$\Rightarrow \lambda = -\frac{2k}{2} \pm \frac{\sqrt{4k^2 - 4(k^2+k^{-2})}}{2}$

$\Rightarrow \lambda = -k \pm \frac{\sqrt{4k^2 - 4k^2 - 4k^{-2}}}{2}$.

$\Rightarrow \lambda = -k \pm \frac{i\,2/k}{2} = -k \pm i/k$ (CASE-III)

Hence the general sol is.

$y = e^{-kx}(A\cos(x/k) + B\sin(x/k))$

Ans

INITIAL VALUE PROBLEMS

Solve the following problems.

(13) $9y'' + 6y' + y = 0$, $y(0) = 4$, $y'(0) = -13/3$

The characteristic eq is.

$9\lambda^2 + 6\lambda + 1 = 0$.

$\Rightarrow 9\lambda^2 + 3\lambda + 3\lambda + 1 = 0 \Rightarrow 3\lambda(3\lambda+1) + 1(3\lambda+1) = 0$.

$\Rightarrow \lambda = -1/3$ (CASE-II)

Hence the general sol is.

$y = (c_1 + c_2 x)e^{-1/3\,x}$ — (1)

Also $y' = -\frac{c_1}{3}e^{-x/3} - \frac{c_2 x}{3}e^{-x/3} + c_2 e^{(-1/3)x}$.

$\Rightarrow y' = \left(-\frac{c_1}{3} - \frac{c_2 x}{3} + c_2\right)e^{-x/3}$ — (2)

Applying initial conditions.

$\Rightarrow -\frac{13}{3} = -\frac{4}{3} + c_2 \Rightarrow -\frac{13}{3} + \frac{4}{3} = -\frac{9}{3} = -3 = c_2$

$\Rightarrow c_2 = -3$ putting in (1)

$\Rightarrow y = (4-3x)e^{-x/3}$ Ans

(14) $4y'' + 16y' + 17y = 0$, $y(0) = -0.5$, $y'(0) = 1$

The characteristic eq is.

$4\lambda^2 + 16\lambda + 17 = 0$.

$\Rightarrow \lambda = \frac{-16 \pm \sqrt{256 - 4(4)(17)}}{8} = -2 \pm \frac{4i}{8}$

$\Rightarrow \lambda = -2 \pm \frac{1}{2} i$ (CASE – III)

Hence the general solution is -

$y = e^{-2x}\left(A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right)\right)$ —— (1)

Also $y' = -2e^{-2x}\left(A\cos\left(\frac{x}{2}\right) + B\sin\left(\frac{x}{2}\right)\right) + e^{-2x}\left(-\frac{1}{2}A\sin\left(\frac{x}{2}\right) + \frac{1}{2}B\cos\left(\frac{x}{2}\right)\right)$

$= e^{-2x}\left(-2A\cos\left(\frac{x}{2}\right) - 2B\sin\left(\frac{x}{2}\right) - \frac{1}{2}A\sin\left(\frac{x}{2}\right) + \frac{1}{2}B\cos\left(\frac{x}{2}\right)\right)$ —— (2)

Applying initial conditions

—— (3)

$(1) \Rightarrow -0.5 = A \Rightarrow A = -\frac{1}{2}$

$(2) \Rightarrow 1 = -2A + \frac{1}{2}B \Rightarrow 1 = -2\left(-\frac{1}{2}\right) + \frac{1}{2}B$

$\Rightarrow \frac{1}{2}B = 1 - 1 \Rightarrow B = 0$ putting in (1)

$\Rightarrow y = e^{-2x}\left(-\frac{1}{2}\cos\left(\frac{x}{2}\right)\right) \Rightarrow -\frac{1}{2}e^{-2x}\cos\left(\frac{x}{2}\right)$ Ans

$\Rightarrow y = -\frac{1}{2}e^{-2x}\cos\left(\frac{x}{2}\right)$.

Check

$y' = -\frac{1}{2}\left(-2e^{-2x}\cos\left(\frac{x}{2}\right) + -\frac{1}{2}e^{-2x}\sin\left(\frac{x}{2}\right)\right)$

$\Rightarrow y' = e^{-2x}\cos\left(\frac{x}{2}\right) + \frac{1}{4}e^{-2x}\sin\left(\frac{x}{2}\right)$

$\Rightarrow y' = e^{-2x}\left(\cos\left(\frac{x}{2}\right) + \frac{1}{4}\sin\left(\frac{x}{2}\right)\right)$

$\Rightarrow y'' = -2e^{-2x}\left(\cos\frac{x}{2} + \frac{1}{4}\sin\left(\frac{x}{2}\right)\right) + e^{-2x}\left(-\frac{1}{2}\sin\frac{x}{2} + \frac{1}{8}\cos\frac{x}{2}\right)$

$\Rightarrow y'' = e^{-2x}\left(-2\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right)$

$\Rightarrow y'' = e^{-2x}\left(-1.5\cos\frac{x}{2} - \sin\frac{x}{2}\right)$

putting in Q.

$\Rightarrow 4e^{-2x}\left(-1.5\cos\frac{x}{2} - \sin\frac{x}{2}\right) + 16e^{-2x}\left(\cos\frac{x}{2} + \frac{1}{4}\sin\frac{x}{2}\right)$

$+17\left(-\frac{1}{2}e^{-2x}\cos\frac{x}{2}\right)$

$= -7.5e^{-2x}\cos\frac{x}{2} - 4e^{-2x}\sin\frac{x}{2} + 16e^{-2x}\cos\frac{x}{2} + 4e^{-2x}\sin\frac{x}{2}$

$-8.5\cos\frac{x}{2} = 0$

satisfied

(15) $y'' - 25y = 0$, $y(0) = 0$, $y'(0) = 20$.

The characteristic eq is.

$\lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$ (CASE – II)

Hence the general sol is

$y = c_1 e^{-5x} + c_2 e^{5x}$ —— ①

Also $y' = -5c_1 e^{-5x} + 5c_2 e^{5x}$ —— ②

applying initial conditions.

① $\Rightarrow 0 = c_1 + c_2$ — ③

② $\Rightarrow 20 = -5c_1 + 5c_2 \Rightarrow 4 = -c_1 + c_2$ — ④

adding ③ and ④

$\Rightarrow 2c_2 = 4 \Rightarrow c_2 = 2$. putting in ③

$\Rightarrow c_1 = -2$ putting in ①

$\Rightarrow y = -2e^{-5x} + 2e^{+5x}$

Ans

Q16 The characteristic equation is

$\lambda^2 + 0.4\lambda + 0.29 = 0$

$\Rightarrow \lambda = \dfrac{-0.4 \pm \sqrt{0.16 - 4(0.29)}}{2} = \dfrac{-0.4 \pm \sqrt{-1}}{2}$

$\Rightarrow \lambda = -0.2 \pm \frac{1}{2}i$. (CASE III)

Hence the general sol is given by

$y = e^{-0.2x}\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$.

Also $y' = -0.2e^{-0.2x}\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right) + e^{-0.2x}\left(-\frac{1}{2}A\sin\frac{x}{2} + \frac{1}{2}B\cos\frac{x}{2}\right)$

$\Rightarrow y' = e^{-0.2x}\left(-0.2A\cos\frac{x}{2} - 0.2B\sin\frac{x}{2} - \frac{1}{2}A\sin\frac{x}{2} + \frac{1}{2}B\cos\frac{x}{2}\right)$

applying initial conditions.

① $\Rightarrow 1 = e^0(A\cos 0 + B\sin 0) \Rightarrow A = 1$

② $\Rightarrow -1.2 = e^0\left(-0.2A + \frac{1}{2}B\right) \Rightarrow -1.2 = -0.2(1) + \frac{1}{2}B$.

$\Rightarrow -1.2 + 0.2 = \frac{1}{2}B \Rightarrow B = -2$.

putting in ① we get the particular sol as.

$y = e^{-0.2x}\left(\cos\frac{x}{2} - 2\sin\frac{x}{2}\right)$ Ans

⑰ $y'' - y' - 2y = 0$, $y(0) = -4$, $y'(0) = -17$

Sol $\lambda^2 - \lambda - 2 = 0$.

$\Rightarrow \lambda^2 - 2\lambda + \lambda - 2 = 0 \Rightarrow \lambda(\lambda - 2) + 1(\lambda - 2) = 0$.

$\Rightarrow \lambda_1 = 2, \lambda_2 = -1$ CASE-I

Hence

$y = c_1e^{2x} + c_2e^{-1x}$ — ①

Also $y' = 2c_1e^{2x} - 1c_2e^{-1x}$ — ②

applying initial conditions.

① $\Rightarrow -4 = c_1 + c_2$ — ③ , ② $\Rightarrow -17 = 2c_1 - 1c_2$ — ④

xing ③ with 1 and adding with ④

$\Rightarrow -4 = 1c_1 + 1c_2$

$\Rightarrow -17 = 2c_1 - 1c_2$

$-21 = 3c_1 \Rightarrow c_1 = -21/3 = -7$ putting in ③ or ④

$$y = -7e^{2x} + 3e^{-x}$$

Ans

(18) $y'' - 2y' + (4\pi^2 + 1)y = 0$, $y(0) = -2$, $y'(0) = 6\pi - 2$

The characteristic eq is

$\lambda^2 - 2\lambda + 4\pi^2 + 1 = 0$

$\Rightarrow \lambda = \dfrac{2 \pm \sqrt{4 - 4(4\pi^2 + 1)}}{2} = \dfrac{2 \pm \sqrt{4 - 16\pi^2 - 4}}{2}$

$\Rightarrow \lambda = \dfrac{2}{2} \pm \dfrac{\sqrt{-16\pi^2}}{2} = 1 \pm \dfrac{i4\pi}{2} = 1 \pm 2\pi i.$

CASE – III

Hence

$y = e^x(A\cos 2\pi x + B\sin 2\pi x)$ —① (G.S)

Also $y' = e^x(A\cos 2\pi x + B\sin 2\pi x) + e^x(-2\pi A\sin 2\pi x + 2\pi B\cos 2\pi x)$ —②

Applying initial conditions

① $\Rightarrow -2 = A\cos 0 + B\sin 0$

$\Rightarrow A = -2.$

and ② $\Rightarrow 6\pi - 2 = A\cos 0 + 2\pi B\cos 0.$

$\Rightarrow 6\pi - \cancel{2} = -\cancel{2} + 2\pi B$

$\Rightarrow B = 3$

putting values of A and B in ①

$\Rightarrow y = e^x(-2\cos 2\pi x + 3\sin 2\pi x)$. (P.S)

Ans

BOUNDARY VALUE PROBLEMS

Solve the following problems.

(19) $y'' + 4y = 0$, $y(0) = 3$, $y(\pi/2) = -3$

$\lambda^2 + 4 = 0.$

$\Rightarrow \lambda = \pm\sqrt{-4} \Rightarrow \lambda = \pm 2i.$ CASE III.

Hence

$y = e^{0x}(A\cos 2x + B\sin 2x).$

$\Rightarrow y = A\cos 2x + B\sin 2x$ —①

Also $y' = 2A\cos 2x + 2B\sin 2x$ —②

Applying boundary conditions

$\Rightarrow 3 = A\cos 0 + B\sin 0$ $\}\ y(0) = 3$.

$\Rightarrow A = 3$

& $-3 = A\cos(2(\frac{\pi}{2})) + B\sin(2(\frac{\pi}{2}))$

$\Rightarrow -3 = A\cos\pi + B\sin\pi$

$\Rightarrow -3 = -A \Rightarrow A = 3$

we see that it yields no conditions for "B"

$\therefore$ The particular solution is

$$\boxed{y = 3\cos 2x + B\sin 2x}$$ Ans

(20) $y'' - 25y = 0$ $\quad y(2) = y(2) = \cosh 10$

The characteristic eq is.

$\lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$ (CASE − I)

Hence $y = c_1 e^{5x} + c_2 e^{-5x}$ —— ①.

Applying boundary condition

$\cosh 10 = c_1 e^{-10} + c_2 e^{10}$ —— ②

Also $\cosh 10 = c_1 e^{10} + c_2 e^{-10}$ —— ③.

x-ing ② with e^{10} and ③ with e^{-10}

$\Rightarrow e^{10}\cosh 10 = c_1 + c_2 e^{20}$ —— ④ $\Rightarrow c_1 = e^{10}\cosh 10 - c_2 e^{20}$

$\Rightarrow e^{-10}\cosh 10 = c_1 + c_2 e^{-20}$ —— ⑤ $\Rightarrow c_1 = e^{-10}\cosh 10 - c_2 e^{-20}$

comparing ④ and ⑤

$\Rightarrow e^{10}\cosh 10 - c_2 e^{20} = e^{-10}\cosh 10 - c_2 e^{-20}$

$\Rightarrow c_2 e^{20} - c_2 e^{-20} = e^{10}\cosh 10 - e^{-10}\cosh 10$.

$\Rightarrow c_2 = \dfrac{\cosh 10(e^{10} - e^{-10})}{(e^{20} - e^{-20})} \Rightarrow c_2 = 0.5$.

putting in ② or ③

③ $\Rightarrow c_1 e^{10} = \cosh 10 - c_2 e^{-10}$

$\Rightarrow c_1 = \dfrac{\cosh 10 - 0.5e^{-10}}{e^{10}} = 0.5$.

putting values of c_1 and c_2 in ①.

$\Rightarrow y = \frac{1}{2}e^{5x} + \frac{1}{2}e^{-5x}$.

Ans

(21) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y(\pi/2) = 0$.

The characteristic equation is:

$\lambda^2 + 2\lambda + 2 = 0$

$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \frac{\sqrt{-4}}{2}$

$\Rightarrow \lambda = -1 \pm i$. (CASE – III)

Hence

$y = e^{-x}(A\cos x + B\sin x)$ —— (1).

Applying boundary conditions.

(1) $\Rightarrow 1 = A$ for $y(0) = 1$.

(1) $\Rightarrow 0 = B$ for $y(\pi/2) = 0$.

putting in (1)

$\Rightarrow y = e^{-x}\cos x$.

Ans.

(22) $3y'' - 8y' - 3y = 0$ $y(-3) = 1$, $y(3) = 1/e^2$.

The characteristic eq is.

$3\lambda^2 - 8\lambda - 3 = 0$.

$\Rightarrow 3\lambda^2 - 9\lambda + \lambda - 3 = 0 \Rightarrow 3\lambda(\lambda - 3) + 1(\lambda - 3) = 0$

$\Rightarrow \lambda_1 = -1/3$, $\lambda_2 = 3$ (CASE – I)

Hence.

$y = c_1 e^{-x/3} + c_2 e^{3x}$ —— (1).

Applying boundary condition.

for $y(-3) = 1$.

(1) $\Rightarrow 1 = c_1 e^{+3/3} + c_2 e^{3(-3)}$

$\Rightarrow 1 = c_1 e^{+1} + c_2 e^{-9}$ —— (1)

for $y(3) = 1/e^2$.

(1) $\Rightarrow \frac{1}{e^2} = c_1 e^{-1} + c_2 e^{+9}$ —— (2).

xing (1) with e^{-1} and (2) with e^{1}.

(1) $\Rightarrow e^{-1} = c_1 + c_2 e^{-10}$ —— (3)

(2) $\Rightarrow \frac{1}{e^{2-1}} = c_1 + c_2 e^{10} \Rightarrow \frac{1}{e} = c_1 + c_2 e^{10}$ —— (4)

Comparing (3) and (4)

$\Rightarrow e^{-1} - c_2 e^{-10} \ldots c_2 e^{10} \Rightarrow c_2(e^{10} - e^{-10}) = \frac{1}{e} - \frac{1}{e}$

$\Rightarrow c_2 = 0$. putting in ③ or ④

① $\Rightarrow e^{-1} = c_1 + (0)e^{-10}$ $\Rightarrow c_1 = 1/e$.

putting values of c_1 and ② in ①

$\Rightarrow y = \frac{1}{e} e^{-x/3} = e^{-x/3-1}$.

<u>Check</u>

$y = e^{-x/3-1} \Rightarrow y' = -\frac{1}{3} e^{(-x/3-1)} \Rightarrow y'' = \frac{1}{9} e^{(-x/3-1)}$

putting in Question.

$3\left(\frac{1}{9}\right)e^{(-x/3-1)} - 8\left(-\frac{1}{3}\right)e^{(-x/3-1)} - 3e^{(-x/3-1)}$

$= \frac{1}{3} e^{(-x/3-1)} + \frac{8}{3} e^{(-x/3-1)} - 3e^{(-x/3-1)}$

$= 3e^{(-x/3-1)} - 3e^{(-x/3-1)} = 0$ <u>satisfies</u>